The three-pass regression filter: A new approach to forecasting using many predictors

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\begin{abstract}
We forecast a single time series using many predictor variables with a new estimator called the three-pass regression filter (3PRF). It is calculated in closed form and conveniently represented as a set of ordinary least squares regressions. 3PRF forecasts are consistent for the infeasible best forecast when both the time dimension and cross section dimension become large. This requires specifying only the number of relevant factors driving the forecast target, regardless of the total number of common factors driving the cross section of predictors. The 3PRF is a constrained least squares estimator and reduces to partial least squares as a special case. Simulation evidence confirms the 3PRF’s forecasting performance relative to alternatives. We explore two empirical applications: Forecasting macroeconomic aggregates with a large panel of economic indices, and forecasting stock market returns with price–dividend ratios of stock portfolios.
\end{abstract}

1. Introduction

A common interest among economists and policymakers is harnessing vast predictive information to forecast important economic aggregates like national product or stock market value. However, it can be difficult to use this wealth of information in practice. If the predictors number near or more than the number of observations, the standard ordinary least squares (OLS) forecaster is known to be poorly behaved or nonexistent.\textsuperscript{1}

How then does one effectively use vast predictive information? A solution well known in the economics literature views the data as generated from a model in which latent factors drive the systematic variation of both the forecast target, \( \mathbf{y} \), and the matrix of predictors, \( \mathbf{X} \). In this setting, the best prediction of \( \mathbf{y} \) is infeasible since the factors are unobserved. As a result, a factor estimation step is required. The literature's benchmark method extracts factors that are significant drivers of variation in \( \mathbf{X} \) and then uses these to forecast \( \mathbf{y} \).

Our procedure springs from the idea that the factors that are relevant to \( \mathbf{y} \) may be a strict subset of all the factors driving \( \mathbf{X} \). Our method, called the three-pass regression filter (3PRF), selectively identifies only the subset of factors that influence the forecast target while discarding factors that are irrelevant for the target but that may be pervasive among predictors. The 3PRF has the advantage of being expressed in closed form and virtually instantaneous to compute.

This paper makes four main contributions. The first is to develop asymptotic theory for the 3PRF. We begin by proving that the estimator converges in probability to the infeasible best forecast in the (simultaneous) limit as cross section size \( N \) and time series dimension \( T \) become large. This is true even when variation in predictors is dominated by target-irrelevant factors. We then derive the limiting distributions for the estimated forecasts and predictive coefficients, and provide consistent estimators of asymptotic covariance matrices that can be used to perform inference. The second contribution of the paper is to verify the

\textsuperscript{1} See Huber (1973) on the asymptotic difficulties of least squares when the number of regressors is large relative to the number of data points.

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finite sample accuracy of our asymptotic theory through Monte Carlo simulations.

We also show that the method of partial least squares (PLS) is a special case of the 3PRF. Like partial least squares, the 3PRF can use the forecast target to discipline its dimension reduction. This emphasizes the covariance between predictors and target in the factor estimation step. But unlike PLS, the 3PRF also allows the econometrician to select additional disciplining variables, or proxies, on the basis of economic theory. Furthermore, because it is a special case of our methodology, the asymptotic theory we develop for the 3PRF applies directly to partial least squares. Recently Groen and Kapetanios (2009) showed the consistency of PLS under sequential N, T limits, while our approach proves consistency in the less restrictive simultaneous N, T limit. Those authors do not derive limiting distributions as we do here and so, to the best of our knowledge, our joint N and T asymptotics are new results to the PLS literature.

In our third contribution, we compare the 3PRF to other methods in order to illustrate the source of its improvement in forecasting performance. The economics literature has relied mainly on principal component regression (PCR) for forecasting problems involving many predictors, exemplified by Stock and Watson (1998, 2000a,b, 2006, 2012), Formi and Reichlin (1996, 1998), Bai and Ng (2002, 2006, 2008), Bai (2003) and Boivin and Ng (2006), among others.2 Like the 3PRF, PCR can be calculated instantaneously for virtually any N and T. Stock and Watson’s key insight is to condense information from the large cross section into a small number of predictive indices before estimating a linear forecast. PCR condenses the cross section according to covariance within the predictors. This identifies the factors driving the panel of predictors, some of which may be irrelevant for the dynamics of the forecast target, and uses those factors to forecast.

In contrast, the 3PRF condenses the cross section according to covariance with the forecast target. PCR must estimate all common factors among predictors to achieve consistency, including those that are irrelevant for forecasting. The 3PRF need only estimate the relevant factors, which are always less than or equal to the total number of factors required by PCR. While this difference is innocuous in large samples, it can be a crucial consideration in small samples.

We are not the first to investigate potential improvements upon PCR factor-based forecasts. Doz et al. (2012) propose quasi-maximum likelihood factor estimation as an alternative to PCR. Bai and Ng (2008) propose statistical thresholding rules that drop variables found to contain irrelevant information, building on the insights in Boivin and Ng (2006). In a similar vein, De Mol et al. (2008) propose Bayesian shrinkage methods. Thresholding and shrinkage methods are especially useful when relevant information is non-pervasive and confined to a subset of predictors. This does not solve the problem of pervasive irrelevant information among predictors. Our approach explicitly allows for both relevant and irrelevant pervasive factors.

The final contribution of the paper is to provide empirical support for the 3PRF’s strong forecasting performance in simulations and two separate empirical applications. We compare 3PRF to PCR, thresholding methods of Bai and Ng (2008), shrinkage methods of De Mol et al. (2008), and the factor analytic approach of Doz et al. (2012). Simulations show that the 3PRF often outperforms alternatives across a variety of factor model specifications. In empirical applications, we find that the 3PRF is a successful predictor of macroeconomic aggregates and equity market returns, and typically outperforms alternative methods.

The paper is structured as follows. Section 2 defines the 3PRF and proves its asymptotic properties. Section 3 reinterprets the 3PRF as a constrained least squares solution, then compares and contrasts it with partial least squares. Section 4 explores the finite sample performance of the 3PRF and other methods in Monte Carlo experiments. Section 5 reports empirical results for 3PRF and other methods’ forecasts in asset pricing and macroeconomic applications. All proofs and supporting details are placed in the Appendix.

2. The three-pass regression filter

2.1. The estimator

There are several equivalent approaches to formulating our procedure, each emphasizing a related interpretation of the estimator. We begin with what we believe to be the most intuitive formulation of the filter, which is the sequence of OLS regressions that gives the estimator its name.

First we establish the environment wherein we use the 3PRF. There is a target variable which we wish to forecast. There exist many predictors which may contain information useful for predicting the target variable. The number of predictors N may be large and number near or more than the available time series observations T, which makes OLS problematic. Therefore we look to reduce the dimension of predictive information, and to do so we assume the data can be described by an approximate factor model. In order to make forecasts, the 3PRF uses proxies: These are variables, driven by the factors (and as we emphasize below, driven by target-relevant factors in particular), which we show are always available from the target and predictors themselves, but may alternatively be supplied to the econometrician on the basis of economic theory. The target is a linear function of a subset of the latent factors plus some unforecastable noise. The optimal forecast therefore comes from a regression on the true underlying relevant factors. However, since these factors are unobservable, we call this the infeasible best forecast.

We write y for the T × 1 vector of the target variable time series from 2, 3, . . . , T + 1.4 Let X be the T × N matrix of predictors, X = (x1, x2, . . . , xT)′ that have been standardized to have unit time series variance. Note that we are using two different typefaces to denote the N-dimensional cross section of predictors observed at time t (x′ t), and the T-dimensional time series of the ith predictor (x′ t). This is to distinguish the time series of predictors from the cross section of predictors in Table 1. We denote the T × L matrix of proxies as Z, which stacks period-by-period proxy data as Z = (z1, z2, . . . , zT)′. We make no assumption on the relationship between N and T but assume L ≪ min(N, T) in the spirit of dimension reduction. We provide additional details regarding the data generating processes for y, X and Z in Assumption 1.

With this notation in mind, the 3PRF’s regression-based construction is defined in Table 1. The first pass runs N separate time series regressions, one for each predictor. In these first pass regressions, the predictor is the dependent variable, the proxies are the regressors, and the estimated coefficients describe the sensitivity of the predictor to factors represented by the proxies. As we show later, proxies need not represent specific factors and

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2 The model investigated by Forni et al. (2000, 2004, 2005) concentrates on a frequency domain approach.

3 We also demonstrate that the performance of 3PRF is robust to cases where relevant information is non-pervasive—that is, when only a subset of predictors have non-zero loadings on the relevant factors.

4 Nothing prevents us from generalizing this to consider direct forecasts of y_{t+h} for h ∈ {1, 2, ..., T}—the theory is identical. For exposition’s sake we deal only with y_{t+1}, knowing that t + 1 could instead be t + h but everything that follows would still hold.
may be measured with noise. The important requirement is that their common components span the space of the target-relevant factors.

The second pass uses the estimated first-pass coefficients in $T$ separate cross section regressions. In these second pass regressions, the predictors are again the dependent variable while the first-pass coefficients $\hat{\phi}_t$ are the regressors. Fluctuations in the latent factors cause the cross section of predictors to fan out and compress over time. First-stage coefficient estimates map the cross-sectional distribution of predictors to the latent factors. Second-stage cross section regressions use this map to back out estimates of the factors at each point in time.

We then carry forward the estimated second-pass predictive factors $\hat{F}_t$ to the third pass. This is a single time series forecasting regression of the target variable $y_{t+1}$ on the second-pass estimated predictive factors $\hat{F}_t$. The third-pass fitted value $\hat{\beta}_0 + \hat{F}_t \hat{\beta}$ is the 3PRF time $t$ forecast. Because the first-stage regression takes an error-in-variables form, second-stage regressions produce an estimate for a unique but unknown rotation of the latent factors. Since the relevant factor space is spanned by $\hat{F}_t$, the third-stage regression delivers consistent forecasts.

An alternative representation for the 3PRF is the one-step closed form:

$$\hat{y} = \hat{v}_t \hat{y} + \hat{f}_t X W_{xZ} \left(W_{xZ} S_{xZ} W_{xZ}\right)^{-1} W_{xZ} s_{xy}$$

where $\hat{f}_t = I_t - \frac{1}{\hat{v}_t} I_t$ for $I_t$ the $T$-dimensional identity matrix and $\hat{v}_t$ the T-vector of ones ($J_n$ is analogous), $\hat{y} = \hat{v}_t^T y / T$, $W_{xZ} \equiv J_n X' J_n Z$, $S_{xZ} \equiv X' J_n S_{xZ}$ and $s_{xy} \equiv X' J_n y$ matrices enter because each regression pass is run with a constant.

The closed form is central to the theoretical development that follows. Nonetheless, the regression-based procedure in Table 1 remains useful for two reasons. First, in practice (particularly with many predictors) one often faces unbalanced panels and missing data. The 3PRF as described in Table 1 easily handles these difficulties. Second, it is useful for developing intuition behind the procedure and for understanding its relation to partial least squares.

We can rewrite the forecast as

$$\hat{y} = \hat{v}_t \hat{y} + \hat{F} \hat{\beta}$$

$$\hat{\beta} = S_{xZ} \left(W_{xZ}^T S_{xZ}\right)^{-1} W_{xZ} X'$$

where $S_{xZ} \equiv X' J_n Z$. Here we interpret $\hat{F}$ as our predictive factor and $\hat{\beta}$ the predictive coefficient on that factor. Since we have used the $N$ predictors to construct a $L$-dimensional predictive factor, the 3PRF reduces the dimension of the forecasting problem.

Alternatively, we can rewrite the forecast

$$\hat{y} = \hat{v}_t \hat{y} + \hat{f}_t X \hat{\alpha}$$

$$\hat{\alpha} = W_{xZ} \left(W_{xZ}^T S_{xZ} W_{xZ}\right)^{-1} W_{xZ} s_{xy}$$

interpreting $\hat{\alpha}$ as the predictive coefficient on individual predictors. The regular OLS estimate of the projection coefficient $\alpha$ is $(S_{xZ})^{-1} s_{xy}$. This representation suggests that our approach can be interpreted as a constrained version of least squares (Theorem 8 shows this formally below). We further discuss the properties of these estimators in Sections 2.3 and 2.4 after presenting our assumptions in the next subsection.

2.2. Assumptions

We next detail the modeling assumptions that provide a groundwork for developing asymptotic properties of the 3PRF.

**Assumption 1 (Factor Structure).** The data are generated by the following:

$$x_t = \phi_0 + \Phi f_t + \epsilon_t \quad y_{t+1} = \beta_0 + \beta' f_t + \eta_{t+1} \quad z_t = \lambda_0 + \Lambda f_t + \omega_t \quad X = \phi_0 + \Phi' \epsilon + \eta$$

$$y = \beta_0 + \beta' X + \epsilon$$

where $f_t = (f_t', g_t', \phi)$, $\Phi = (\Phi_T, \Phi_Z)$, $\Lambda = (\Lambda_T, \Lambda_Z)$, and $\beta = (\beta_T', \beta_Z')$ with $|\beta_T| > 0$, $K_T > 0$ is the dimension of vector $f_t$, $K_Z > 0$ is the dimension of vector $g_t$, $L$ is the dimension of vector $\epsilon$, $f_t \in (0 < L \leq \min(N, T))$, and $K = K_T + K_Z$.

**Assumption 1** defines the factor structure. The target’s factor loadings $(\beta_T, 0')'$ allow the target to depend on a strict subset of the factors driving the predictors. We refer to this subset as the relevant factors, which are denoted $f_t$. In contrast, irrelevant factors, $g_t$, do not influence the forecast target but may drive the cross section of predictive information $x_t$. The proxies $z_t$ are driven by factors and proxy noise.

**Assumption 2 (Factors, Loadings and Residuals).** Let $M < \infty$. For any $i, s, t$

1. $\|F_{i}\|_4 < M$, $T^{-1} \sum_{s=1}^{T} F_{s} \xrightarrow{T \to \infty} m$ and $T^{-1} F_{i} F_{i} \xrightarrow{T \to \infty} \Delta_{f}$
2. $\|\phi_{i}\|_4 \leq M$, $N^{-1} \sum_{i=1}^{N} \phi_{i} \xrightarrow{N \to \infty} \phi$, $N^{-1} \phi_{i} \phi_{i} \xrightarrow{N \to \infty} \phi$ and $N^{-1} \phi_{i} \phi_{i} \xrightarrow{N \to \infty} \lambda_{i}^{(1)}$
3. $\|\epsilon_{1}\|_{2} = 0$, $\|\epsilon_{2}\|_{2} \leq M$
4. $\|\omega_{t}\|_{2} = 0$, $\|\omega_{t}\|_{4} \leq M$, $T^{-1/2} \sum_{s=1}^{T} \omega_{s} = O_{p}(1)$ and $T^{-1} \omega_{t} \omega_{t} \xrightarrow{T \to \infty} \Delta_{\omega}$
5. $\epsilon_{i}(\eta_{i})_{1} \equiv \epsilon(\eta_{i+1}|y_{f_{i}}, f_{i-1}, f_{i-2}, \ldots) = 0$, $\epsilon(\eta_{i+1}) \leq M$, and $\eta_{i+1}$ is independent of $\phi_{0}(m)$ and $\epsilon_{1}$.

Since $\eta_{t+1}$ is a martingale difference sequence with respect to all information known at time $t$, $\beta_0 + \beta' f_t$ gives the best time $t$ forecast. But it is infeasible since the relevant factors $f_t$ are unobserved.

We require factors and loadings to be cross-sectionally regular in that they have well-behaved covariance matrices for large $T$ and $N$, respectively. **Assumption 2.4** does not exist in the work of Stock and Watson or Bai and Ng, and is required because the 3PRF uses proxies to extract factors. We bound the moments of proxy noise $\omega_t$ in the same manner as the bounds on factor moments.

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5 If coefficients were observable, this mapping would be straightforward since factors could be directly estimated each period with cross section regressions of predictors on the loadings. While the loadings in our framework are unobservable, the same intuition for recovering the factor space applies to our cross section regressions. The difference is that we use estimated loadings as stand-ins for the unobservable true loadings.

6 $\|\phi_{i}\|_{4}$ can replace $\|\phi_{i}\|_{4} \leq M$ if $\phi_{i}$ is non-stochastic.
Assumption 3 (Dependence). Let \( x(m) \) denote the \( m \)th element of \( x \). For \( M < \infty \) and any \( i, j, t, s, m_i, m_j \)

1. \( E(\varepsilon_{it}\varepsilon_{is}) = \sigma_{ij,t}, |\sigma_{ij,t}| \leq \tau_{js}, \) and \( |\sigma_{ij,t}| \leq \tau_{os}, \) and
   (a) \( N^{-1} \sum_{t=1}^{N} \hat{\theta}_{it} \leq M \)
   (b) \( T^{-1} \sum_{t=1}^{T} \tau_{is} \leq M \)
   (c) \( N^{-1} \sum_{t=1}^{N} |\hat{\theta}_{it}| \leq M \)
   (d) \( N^{-1}T^{-1} \sum_{t=1}^{N} \tau_{is} \leq M \)

2. \( E(\varepsilon_{it}^2) = \sigma_{ii,t}, |\sigma_{ii,t}| \leq \tau_{it}, \) and \( |\sigma_{ii,t}| \leq \tau_{st}, \)

3. \( E(\varepsilon_{it}\varepsilon_{is}) = \sum_{t=1}^{T} \sum_{s=1}^{N} \varepsilon_{it} \varepsilon_{is} = E(\varepsilon_{it}^2) \leq M \)

4. \( \sum_{t=1}^{T} \sum_{s=1}^{N} \varepsilon_{it} \varepsilon_{is} \leq M \)

Note that Assumptions 2.4, 3.3, 3.4 and 6 are the only conditions involving the proxy variables. We prove in Theorem 7 that automatic proxies, which are generally constructable using \( X \) and \( y \) are guaranteed to satisfy these proxy assumptions.

With these assumptions in place, we derive the asymptotic properties of the three-pass regression filter. Our proofs build upon the seminal theory of Stock and Watson (2002a), Bai (2003) and Bai and Ng (2002, 2006). Portions of auxiliary lemmas in the appendix draw directly from convergence results proved in these previous papers. Theorems reported in the main text are our central new results. In order to keep our theoretical development self-contained, we catalogue all theoretical results in the Appendix.

2.3. Consistency

Theorem 1. Let Assumptions 1–6 hold. The three-pass regression filter forecast is consistent for the infeasible best forecast, \( \hat{y}_{it+1} \rightarrow \beta_0 + F_i \beta \).

Theorem 1 says that the 3PRF is consistent so that for large \( N \) and \( T \) the difference between this feasible forecast and the infeasible best vanishes. This and our other asymptotic results are based on simultaneous and \( T \) and \( N \) limits. As discussed by Bai (2003), the existence of a simultaneous limit implies the existence of coinciding sequential and pathwise limits, but the converse is not true. We refer readers to that paper for a more detailed comparison of these three types of joint limits.

The Appendix also establishes probability limits of first pass time series regression coefficients \( \phi_i \), second pass cross section factor estimates \( F_i \), and third stage predictive coefficients \( \beta \). While primarily serving as intermediate inputs to the proof of Theorem 1, in certain applications these probability limits are useful in their own right. We refer interested readers to Lemmas 3 and 4 in the Appendix.

The estimated loadings on individual predictors, \( \hat{\alpha} \), play an important role in the interpretation of the 3PRF. The next theorem provides the probability limit for the loading on each predictor.

Theorem 2. Let \( \hat{\alpha}_i \) denote the \( i \)th element of \( \hat{\alpha} \), and let Assumptions 1–6 hold. Then for any \( i \)

\[
N \hat{\alpha}_i \rightarrow \frac{p}{T,N \rightarrow \infty} \left( \phi_i - \bar{\phi} \right) \beta.
\]

The coefficient \( \alpha \) maps underlying factors to the forecast target via the observable predictors. As a result the probability limit of \( \alpha \) is a product of the loadings of \( X \) and \( y \) on the relevant factors \( F \). This arises from the interpretation of \( \alpha \) as a constrained least squares coefficient estimate, which we elaborate on in the next section. Note that \( \alpha \) is multiplied by \( N \) in order to derive its limit. This is because the dimension of \( \alpha \) grows with the number of predictors. As \( N \) grows, the predictive information in \( F \) is spread across a larger number of predictors so each predictor’s contribution approaches zero. Standardizing by \( N \) is necessary to identify the non-degenerate limit.

What distinguishes these results from previous work using PCR is the fact that the 3PRF uses only as many predictive factors as the number of factors relevant to \( y_{t+1} \). In contrast, the PCR forecast is asymptotically efficient when there are as many predictive factors as the total number of factors driving \( X \). (Stock and Watson, 2002a). This distinction is especially important when the number of relevant factors is strictly less than the number of total factors in the predictor data and the target-relevant principal components are dominated by other components in \( X \). In particular, if the factors driving the target are weak in

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[7] Stock and Watson (2002a) summarize this point (we have replaced their symbols with ours notation):

Because \( \Phi F = \Phi R \Phi^{-1} F \) for any nonsingular matrix \( R \), a normalization is required to uniquely define the factors. Said differently, the model with factor loadings \( \Phi F \) and factors \( F \) is observationally equivalent to the model with factor loadings \( \Phi \) and factors \( F \). Assumption 5 restricts \( R \) to be orthonormal and... restricts \( R \) to be a diagonal matrix with diagonal elements of \( \pm 1 \). We further discuss our normalization assumption in Appendix A.7. There we prove that a necessary condition for convergence to the infeasible best forecast is that the number of relevant proxies equals the number of relevant factors.
the sense that they contribute a only small fraction of the total variability in the predictors, then principal components may have difficulty identifying them. Said another way, there is no sense in which the method of principal components is assured to first extract predictive factors that are relevant to \( y_{t+1} \). This point has in part motivated recent econometric work on thresholding (Bai and Ng, 2008) and shrinking (Stock and Watson, 2012) principal components for the purposes of forecasting.

On the other hand, the 3PRF identifies exactly those relevant factors in its second pass factor estimation. This step extracts leading indicators—estimated factors that are specifically valuable for forecasting a given target. To illustrate how this works, consider the special case in which there is only one relevant factor, and the sole proxy is the target variable \( y_{t+1} \) itself. We refer to this case as the target-proxy three-pass regression filter. The following corollary is immediate from Theorem 1.

**Corollary 1.** Let Assumptions 1–5 hold with the exception of Assumptions 2.4, 3.3 and 3.4. Additionally, assume that there is only one relevant factor. Then the target-proxy three-pass regression filter forecaster is consistent for the infeasible best forecast.

**Corollary 1** holds regardless of the number of irrelevant factors driving \( X \) and regardless of where the relevant factor stands in the principal component ordering for \( X \). Compare this to PCR, whose first predictive factor is ensured to be the one that explains most of the covariance among \( x_t \), regardless of that factor’s relationship to \( y_{t+1} \). Only if the relevant factor happens to also drive most of the variation within the predictors does the first component achieve the infeasible best. It is in this sense that the forecast performance of the 3PRF is robust to the presence of irrelevant factors.

### 2.4. Asymptotic distributions

Not only is the 3PRF consistent for the infeasible best forecast, each forecast has a normal asymptotic distribution. We first derive the asymptotic distribution for \( \hat{\alpha} \) since this is useful for establishing the asymptotic distribution of forecasts.

**Theorem 3.** Under Assumptions 1–6, as \( N, T \to \infty \) we have

\[
\sqrt{T} \left( \hat{\alpha}_t - \bar{\alpha}_t \right) \overset{d}{\to} \mathcal{N}(0,1)
\]

where \( \alpha_t^2 \) is the \( t \)th diagonal element of \( \hat{\text{Var}}(\hat{\alpha}) = \Omega_{\alpha}(\frac{1}{T} \sum_i \hat{\eta}_{i,t}^2 (X_t - \bar{X})(X_t - \bar{X})' \Omega_{\alpha}', \hat{\eta}_{t+1} \) is the estimated 3PRF forecast error, \( \bar{\alpha}_t \equiv S_{\alpha} \hat{\beta} \), where \( S_{\alpha} \) is the \( t \)th column of the vector \( \Omega_{\alpha}/\hat{\beta} \) and

\[
\Omega_{\alpha} = J_N \left[ 1 - T'X'J'Z \right] \left( T^{-3}N^{-2}W_{x2}S_{\alpha}W_{x2} \right)^{-1} \left( N^{-1}T^{-2}W_{x2}X'J'F \right),
\]

and

\[
\Omega_{\alpha} = J_N \left( \frac{1}{T} S_{\alpha} \right) \left[ \frac{1}{F'J'Z} T W_{x2} S_{\alpha} W_{x2} \right]^{-1} \left( \frac{1}{N} W_{x2} \right).
\]

While Theorem 2 demonstrates that \( \hat{\alpha} \) may be used to measure the relative forecast contribution of each predictor, Theorem 3 offers a distribution theory, including feasible \( t \)-statistics, for inference. The \( G_{\alpha} \) matrix appears here because the factors are only identified up to an orthonormal rotation.

From here, we derive the asymptotic distribution of the 3PRF forecasts.

**Theorem 4.** Under Assumptions 1–6, as \( N, T \to \infty \) we have

\[
\sqrt{T} \left( \hat{y}_{t+1} - E_{\hat{y}_{t+1}} \right) \overset{d}{\to} \mathcal{N}(0,1)
\]

where \( E_{\hat{y}_{t+1}} = \beta_0 + \beta \hat{F} \) and \( Q_t^2 \) is the \( t \)th diagonal element of \( \frac{1}{T} J_t X \text{Var}(\hat{\alpha}) X' \hat{J}_t \).

This result shows that besides being consistent for the infeasible best forecast \( E_{\hat{y}_{t+1}} \equiv \beta_0 + \beta \hat{F} \), the 3PRF forecast is asymptotically normal and provides a standard error estimator for constructing forecast confidence intervals. A subtle but interesting feature of this result is that we only need the asymptotic variance of individual predictor loadings \( \text{Var}(\hat{\alpha}) \) for the prediction intervals. This differs from the confidence intervals of PCR forecasts in Bai and Ng (2006), which require an estimate of the asymptotic variance for the predictive factor loadings (the analogue of our \( \text{Var}(\hat{\beta}) \) below) as well as an estimate for the asymptotic variance of the fitted latent factors, \( \text{Var}(\hat{F}) \). Unlike PCR, our framework allows us to represent loadings on individual predictors in a convenient algebraic form, \( \hat{\alpha} \). Inspection of \( \hat{\alpha} \) reveals why variability in both \( \hat{\beta} \) and \( \hat{F} \) is captured by \( \text{Var}(\hat{\alpha}) \).

Next, we provide the asymptotic distribution of predictive loadings on the latent factors and a consistent estimator of their asymptotic covariance matrix.

**Theorem 5.** Under Assumptions 1–6, as \( N, T \to \infty \) we have

\[
\sqrt{T} \left( \hat{\beta} - G_{\beta} \beta \right) \overset{d}{\to} \mathcal{N}(0, \Sigma_{\beta})
\]

where \( \Sigma_{\beta} = \Sigma_{\beta}^{-1} \Gamma_{\beta} \Sigma_{\beta}^{-1} \) and \( \Sigma_{\beta} = \Lambda_{\Delta F} \Lambda' + \Lambda_{\Delta Y} \). Furthermore,

\[
\text{Var}(\hat{\beta}) = \left( T^{-1} \hat{F} \hat{F}' \right)^{-1} T^{-1} \sum_i \hat{\eta}_{i,t+1} (\hat{F}_i - \hat{\mu}_f) (\hat{F}_i - \hat{\mu}_f)'
\]

\[
\times \left( T^{-1} \hat{F} \hat{F}' \right)^{-1}
\]

is a consistent estimator of \( \Sigma_{\beta} \). \( G_{\beta} \) is defined in the Appendix.

We also derive the asymptotic distribution of the estimated relevant latent factor rotation.

**Theorem 6.** Under Assumptions 1–6, as \( N, T \to \infty \) we have for every \( t \)

(i) if \( \sqrt{N}/T \to 0 \), then

\[
\sqrt{T} \left( \hat{F}_t - (H_0 + HF_t) \right) \overset{d}{\to} \mathcal{N}(0, \Sigma_{\beta})
\]

(ii) if \( \lim \inf \sqrt{N}/T \geq \tau > 0 \), then

\[
T \left[ \hat{F}_t - (H_0 + HF_t) \right] \overset{d}{\to} \mathcal{O}_p(1)
\]

where \( \Sigma_{\beta} = \left( \Lambda_{\Delta Y} \Lambda' + \Lambda_{\Delta Y} \right) \left( \Lambda_{\Delta F} \Lambda' + \Lambda_{\Delta Y} \right)^{-1} \Lambda_{\Delta F} \Gamma_{\beta} \Lambda_{\Delta F} \Lambda' + \Lambda_{\Delta Y} \)

The matrices \( G_{\beta} \) and \( H \) are present since we are in effect estimating a vector space. Quoting Bai and Ng (2006), Theorems 5 and 6 in fact “pertain to the difference between [\( \hat{F}_t/\hat{\beta} \)] and the space spanned by [\( \hat{F}_t/\hat{\beta} \)].” Note that we do not provide an estimator the asymptotic variance of \( \hat{F} \). While under some circumstances such an estimator is available, this is not generally the case. In particular, when there exist irrelevant factors driving the predictors, the 3PRF only estimates the relevant factor subspace. This complicates the construction of a consistent estimator of \( \text{Var}(\hat{F}) \). Estimators for the asymptotic variance of \( \hat{\alpha} \), \( \hat{\beta} \) and \( \hat{y}_{t+1} \) do not confront this difficulty for reasons discussed following Theorem 4.

2.5. Proxy selection

The formulation of the filter, and its success in forecasting even when principal components that dominate cross section variation are irrelevant to the forecast target, relies on the existence of proxies that depend only on target-relevant factors. This begs
the question: Need we make an a priori assumption about the availability of such proxies? The answer is no—there always exist readily available proxies that satisfy the relevance criterion of Assumption 6. They are obtained from an automatic proxy selection algorithm which constructs proxies that depend only upon relevant factors. For now we treat the true number of relevant factors as known, and return to a discussion of statistical criteria for selecting the appropriate number of 3PRF factors in Section 4.2.

### 2.5.2. Theory proxys

The use of automatic proxies in the three-pass filter disciplines dimension reduction of the predictors by emphasizing the covariance between predictors and target in the factor estimation step. The filter may instead be employed using alternative disciplining variables (factor proxies) which may be distinct from the target and chosen on the basis of economic theory or by statistical arguments. Consider a situation in which \( K_f \) is one, so that the target and proxy are given by \( \hat{y}_{t+1} = \beta_0 + \beta f_{t} + \eta_{t+1} \) and \( z_{t} = \lambda_{0} + \lambda f_{t} + \omega_{t} \). Also suppose that the population \( R^2 \) of the proxy equation is substantially higher than the population \( R^2 \) of the target equation.

The forecasts from using either \( z_{t} \) or the target as proxy are asymptotically identical. However, in finite samples, forecasts can be improved by proxying with \( z_{t} \) due to its higher signal-to-noise ratio.\(^8\) To illustrate this point, in Section 5 we consider a macroeconomic application of theory proxies. We find that improved out-of-sample forecasts of inflation come by imposing a dynamic quantity theory of inflation. These forecasts have an attractive feature that they can accurately be described as embodying an economic narrative – that output and money growth fuel price inflation – that could serve to make the forecasts more appealing to policy-makers or institutional investors.

### 3. Related procedures

Comparing our procedure to other methods develops intuition for why the 3PRF produces powerful forecasts. Adding to our earlier comparisons with PCR, this section evaluates the link between the 3PRF and constrained least squares and partial least squares. Importantly, we show that the 3PRF is the constrained least squares estimate of the projection of \( y \) onto \( X \). The constraint impose embodies the assumption that proxies span the relevant factor space.\(^10\) It happens that partial least squares emerges as a special case of the 3PRF using automatic proxies.

#### 3.1. Constrained least squares

Section 2.1 demonstrates that the forecast \( \hat{y}_{t+1} \) may be represented not only in terms of factor loadings (\( \hat{f}_i \)), but equivalently in terms of loadings on individual predictors (\( \hat{\alpha}_i \)). The ith element of coefficient vector \( \hat{\alpha} \) provides a direct statistical description for the forecast contribution of predictor \( x_i \) when it is combined with the remaining \( N-1 \) predictors. In fact, \( \hat{\alpha} \) is an \( N \)-dimensional projection coefficient, and is available when \( N \) is near or even greater than \( T \). This object allows us to address questions that would typically be answered by the multiple regression coefficient in settings where OLS is unsatisfactory. As discussed by Cochrane (2011) in his presidential address to the American Finance Association:

|W|e have to move past treating extra variables one or two at a time, and understand which of these variables are really important. Alas, huge multiple regression is impossible. So the challenge is, how to answer the great multiple-regression question, without actually running huge multiple regressions?\(^9\)

---

8 While we may always recast the system in terms of a single relevant factor \( f_i \), and rotate the remaining factors to be orthogonal to it, this does not generally alleviate the requirement for as many proxies as relevant factors. As we demonstrate in Appendix A.7, this is because rotating the factors necessarily implies a rotation of factor loadings. Taking both rotations into account recovers the original requirement for as many relevant proxies as relevant factors.

9 On the other hand, if theory-motivated proxies are weakly correlated with the true relevant factors, then the 3PRF will break down and fail to identify a meaningful forecasting relationship. This point is raised by Kleibergen and Zhan (2013) in the context of Fama and MacBeth (1973) two-pass regression.

10 Of course, this span can be measured with error in the sense formalized by our assumptions regarding the proxy noise \( \epsilon_i \).
The 3PRF estimator \( \hat{\alpha} \) provides an answer. It is a projection coefficient relating \( y_{t+1} \) to \( x_t \) under the constraint that irrelevant factors do not influence forecasts. That is, the 3PRF forecaster may be derived as the solution to a constrained least squares problem, as we demonstrate in the following proposition.

**Theorem 8.** The three-pass regression filter's implied \( N \)-dimensional predictive coefficient, \( \hat{\alpha} \), is the solution to

\[
\arg \min_{a_0, \alpha} \| y - a_0 - X\alpha \|
\]

subject to \( (I - W_{XZ}(S_{WZ}W_{XZ})^{-1}W_{XZ})\alpha = 0 \). (5)

This solution is closely tied to the original motivation for dimension reduction: The unconstrained least squares forecaster is poorly behaved when \( N \) is large relative to \( T \). The 3PRF’s answer is to impose the constraint in Eq. (5), which exploits the proxies and has an intuitive interpretation. Premultiplying both sides of the equation by \( J_tX \), we can rewrite the constraint as \( (J_tX - J_t\hat{F}\hat{\Phi})\alpha = 0 \). For large \( N \) and \( T \),

\[ J_tX - J_t\hat{F}\hat{\Phi} \approx \varepsilon + (F - \mu)(I - S_{Y})\Phi \]

which follows from Lemma 6 in the Appendix. Because the covariance between \( \alpha \) and \( \varepsilon \) is zero by the assumptions of the model, the constraint simply imposes that the product of \( \alpha \) and the target-irrelevant common component of \( X \) is equal to zero. This is because the matrix \( I - S_{Y} \) selects only the terms in the total common component \( F\Phi \) that are associated with irrelevant factors. This constraint is important because it ensures that factors irrelevant to \( y \) drop out of the 3PRF forecast. It also ensures that \( \hat{\alpha} \) is consistent for the factor model’s population projection coefficient of \( y_{t+1} \) on \( x_t \).

3.2. Partial least squares

The method of partial least squares, or PLS (Wold (1975), described in Appendix A.10), is a special case of the three-pass regression filter. In particular, partial least squares forecasts are identical to those from the 3PRF when (i) the predictors are demeaned and variance-standardized in a preliminary step, (ii) the first two regression passes are run without constant terms and (iii) proxies are automatically selected. As an illustration, consider the case where a single predictive index is constructed from the partial least squares algorithm. Assume, for the time being, that each predictor has been previously standardized to have mean zero and variance one. Following the construction of the PLS forecast given in Appendix A.10 we have

1. Set \( \hat{\phi}_1 = x_t'y_t \) and \( \hat{\phi} = (\hat{\phi}_1, \ldots, \hat{\phi}_N)' \).
2. Set \( \hat{u}_i = x_t'\hat{\phi} \) and \( \hat{u} = (\hat{u}_1, \ldots, \hat{u}_T)' \).
3. Run a predictive regression of \( y \) on \( \hat{u} \).

Constructing the forecast in this manner may be represented as a one-step estimator

\[ y^{PLS} = XX'y(XX'XX'y)^{-1}yXX'y \]

which upon inspection is identical to the 1-automatic-3PRF forecast when constants are omitted from the first and second passes. Repeating the comparison of 3PRF and PLS when constructing additional predictive factors under conditions (i)-(iii) shows that this equivalence holds more generally.

How do the methodological differences between the auto-proxy 3PRF and PLS embodied by conditions (i)-(iii) affect forecast performance? First, since both methods (like PCR as well) lack scale-invariance, they each work with variance-standardized predictors. For PLS, the demeaning of predictors and omission of a constant in first pass regressions offset each other and produce no net difference versus the auto-proxy 3PRF. The primary difference therefore lies in the estimation of a constant in the second stage cross section regression of the auto-proxy 3PRF. A simple example in the context of the underlying factor model assumptions of this paper helps identify when estimating a constant in cross section regressions is useful. Consider the special case of Assumption 1 in which \( K_f = 1 \) and \( K_r = 1 \), the predictors and factors have mean zero, and the relevant factor’s loadings are known. In this case, \( x_t = \Phi_1u_t + \Phi_2F_t + \mu_t \), and the second stage population regression of \( x_t \) on \( \phi_1 \) including a constant yields a slope estimate of \( \lim_{\mu_t \rightarrow \infty} \hat{f}_t = f_t + g_t \parallel \frac{\operatorname{Var}(\hat{\phi}_1u_t)}{\operatorname{Var}(\phi_1)} \parallel \), which reduces to \( f_t \) by Assumptions 2.2 and 5. The slope estimate omitting the constant is \( \lim_{\mu_t \rightarrow \infty} \hat{f}_t = f_t + g_t \parallel \frac{\operatorname{Var}(\phi_1)}{\operatorname{Var}(\hat{\phi}_1u_t)} \parallel \). This is an error-ridden version of the true target-relevant factor, and thus can produce inferior forecasts. Because PLS is a special case of our methodology, the asymptotic theory we have developed for the 3PRF applies directly to PLS estimates. Our results therefore provide a means of conducting inference when applying PLS. Groen and Kapetanios (2009) proved the consistency of PLS using sequential \( N \), \( T \) limits and weak factor assumptions, but did not derive limiting distributions. To the best of our knowledge, our simultaneous \( N \) and \( T \) asymptotics are new results for the PLS literature.

4. Simulation evidence

4.1. Comparison against alternatives

We conduct Monte Carlo experiments to examine the finite sample accuracy of 3PRF forecasts. Our simulations focus on out-of-sample forecast performance and compare this against five alternative procedures. The first alternative is PCR using the first five principal components (PCR5 henceforth), as advocated by Stock and Watson (2002a, 2012). The second and third are least-angle regression and LASSO versions of the “targeted predictors” approach proposed by Bai and Ng (2008). Here, the \( L_1 \) tuning parameter is adjusted to select a group of 30 targeted predictors, from which five principal components are then extracted and used for forecasting. We call these procedures PCMLAR and PCLAS, respectively. The fourth alternative follows the Bayesian shrinkage approach proposed by De Mol et al. (2008). Shrinkage motivates LAR/LASSO wherein the \( L_1 \) tuning parameter is adjusted to select a group of 10 predictors. We call this procedure 10LAR. Finally, we consider the quasi-maximum likelihood factor analysis approach of extracting five factors. We call this procedure FA. We compare each of those multivariate forecasts to forecasts from single predictive index constructed from the target-proxy 3PRF (denoted 3PRF1).

Our simulations use a range of specifications to examine how performance of the estimators is affected by various data features that may complicate factor extraction and forecasting. These include serial correlation in common factors and serial or cross-sectional correlation in idiosyncratic shocks. We also explore how the strength of the factor structure affects performance.

12 In the Appendix, we report Monte Carlo simulations that evaluate whether the asymptotic distribution theory developed in Section 2 is a good approximation of the finite sample distribution of 3PRF estimates.

13 De Mol et al. (2008) empirical exercise found that roughly ten predictors gave them the best forecast performance, and so we use that specification. Similarly, Bai and Ng (2008) also consider a LAR/LASSO procedure to select a group of five predictors.
estimator relative to a naive forecast based on the target’s historical mean. According to factors in all cases. We used datasets of dimension across estimators using simulated data. We simulated data according to $\beta_f \sim t$ and $\beta_{fg} \sim IIN(0, 1)$ and $\beta_{fu} \sim IIN(0, 1, \Sigma_u)$, with $\Sigma_u$ and $\Sigma_{fg}$ uncorrelated and $K_1 = K_2 = 4$ so that $K = 5$. Parameters of the diagonal matrix $\Sigma_u$ are chosen so that irrelevant factors are dominant, in the sense that they have variances $1.25, 1.75, 2.25$ and 2.75 times larger than the relevant factor. The parameters $\rho_f$ and $\rho_g$ govern serial correlation among factors and take values of

By factor strength, we mean the proportion of variation among predictors that is due to the common factors. Lastly, we consider how the persuasiveness of the factor structure impacts estimator performance, where we define “persuasiveness” as the fraction of predictors with non-zero loadings on common factors.

Table 3 reports the out-of-sample forecasting performance across estimators using simulated data. We simulate data according to Assumption 1 using one relevant factor and four irrelevant factors in all cases. We use data sets of dimensions $N, T = 100$ or $N, T = 200$. For each parameter configuration, we conduct 5000 simulations and report the median out-of-sample forecast percentage $R^2$ for each method. The strength of the factor structure may be “normal” (Panel A), in which the predictors have a median $R^2$ of 30% on the factors, “moderately weak” (Panel B) with $R^2$ of 20%, or “weak” (Panel C) with $R^2$ of 10%. The normal structure is roughly in line with the degree of common variation documented in Stock and Watson’s (2002b) analysis of macroeconomic data, while the weak structure is motivated by Groen and Kapetanios (2009) and Onatski (2012). Because the factor loadings are drawn at random in each simulation, there is variation across predictors in the fraction of their variance explained by the factors. In Panels A–C we simulate a pervasive relevant factor, meaning that all predictors have a non-zero loading on it. In Panel D, we report results when the relevant factor is non-pervasive by imposing that half of the predictors have a loading of zero on the relevant factor, otherwise all predictors have non-zero loadings on all factors. We bold the best median performer for each specification when it outperforms the target’s historical mean. The procedures are described in the text.

---

14 The $R^2$ measure is related to the relative mean squared error (RMSE) statistic according to $R^2 = 1 - \text{RMSE}$. It summarizes the forecast performance of each estimator relative to a naive forecast based on the target’s historical mean.
We calculate degrees of freedom via the Krylov representation method of Kramer and Sugiyama (2011), then use this to compute the Bayesian Information Criterion (BIC). Details of this approach are given in Appendix A.9.

To study the BIC accuracy in our setting we simulate data according to the same data generating processes used in Table 3. Results are reported in Table A.2 and show that the information criterion is typically accurate in selecting the correct number of 3PRF factors. For example, when \( N = T = 200 \) and the true number of factors is equal to one, the average number of factors selected across simulations equals 1.0 in 15 of 27 specifications, and is between 1.0 and 1.3 in 21 of 27. The BIC tends to overestimate the number of factors in smaller samples and when the irrelevant factors and residuals exhibit strong serial correlation. But even when too many 3PRF factors are selected, the method achieves powerful out-of-sample forecasting performance and is typically close to the \( R^2 \) achieved by the one-factor 3PRF. Further details are discussed in Appendix A.9.

5. Empirical evidence

Here we report the results of two separate empirical investigations. In the first empirical investigation, we forecast macroeconomic aggregates using a well-known panel of quarterly macroeconomic variables. In the second, we use a factor model to relate individual assets’ price–dividend ratios to market returns.

5.1. Forecasting macroeconomic aggregates

We examine the forecastability of macroeconomic aggregates based on a large number of potential predictor variables. To maintain comparability to the literature, we take as our predictors a set of 108 macroeconomic variables compiled by Stock and Watson (2002b) updated through the end of 2009. Any variable that we eventually target is removed from the set of predictors. We focus attention on pseudo out-of-sample forecasting exercises described in detail in Appendix A.11 and following Bai and Ng (2008) and Stock and Watson (2012). We focus attention on macroeconomic aggregates that receive considerably attention in the literature and policy-making circles. For 3PRF, PCR and FA we consider single factor implementations because ours and Bai and Ng’s (2002) information criteria consistently choose a single factor across forecast targets and training samples.

Table 4 presents our recursive out-of-sample forecasting results. In these macroeconomic data we see a great deal similarity in different procedures’ out-of-sample forecast performance. Even ordinary PCR1 does very well here, often beating more sophisticated procedures involving LAR. The closest competitor is 10LAR
We estimate the extent of market return predictability using 25 log price–dividend ratios of portfolios sorted by market equity and book-to-market ratio. The data is annual over the post-war period 1945–2010 (following Fama and French (1992), see Appendix for details of our data construction). We assume that the predictors take the form

\[ \text{price}_t = \beta_0 + \beta_1 Z_1 + \epsilon_t, \]

where the target takes the form

\[ \text{log price}_t = \beta_0 + \beta_1 F_1 + \epsilon_t. \]

Our out-of-sample analysis here is recursive, as in the case of the previous macroeconomic application, which is common to this literature and described in detail in Appendix A.21 To maintain comparability to our previous macroeconomic results, we begin out-of-sample forecasts in 1985.

We consider the performance of 3PR, PCA, and FA with one or two factors. We also use our Bai and Ng’s (2002) information criteria to estimate the number of factors present in the cross section of value ratios and report those results, as well as the (average) number of factors chosen across all periods of the out-of-sample procedure, for both PCA and FA. Finally, we report the 10LAR procedure of De Mol et al. (2008).22

Table 5 reports market return forecasts and shows that the 3PRF achieves strong out-of-sample performance. The BIC picks one or two factors in most samples, with an average of 1.4. The 3PRF–IC finds an out-of-sample R² of 31.1%, just below the 3PRF2 R² of 36.3%. This performance is significantly higher than what is achieved using PCR, 10LAR or FA.23 In fact, using just the first two PCs results in negative out-of-sample performance, and it requires four or five PCs to extract relevant predictive information and obtain a 27% out-of-sample R². Among dimension reduction techniques, the 3PRF demonstrates the strongest out-of-sample predictive power for aggregate returns.

5.3. Examples of theory proxies for macroeconomic forecasts

Economic implementation of dimension reduction techniques often defy interpretation. They are an amalgamation of different predictors which represent many different economic forces.24 As discussed in Section 2.5.2, the theory-proxy 3PRF provides an applied researcher the opportunity capture some economic interpretability within the dimension reduction procedure. We now provide an example of this approach in the context of inflation forecasting.

Consider the problem of forecasting GDP inflation. Table 4 shows that it is difficult for the previously-considered estimators

22 We do not report the PCLAR or PCLAS procedures since our cross-sectional dimension is 25 and so those procedures coincide with the PCR’s procedure which we implicitly consider in PC–IC.

23 As has been well-documented in the literature, these financial data have a strong factor structure. The first five PCs explain an average of 95.8% of the price–dividend ratios’ variation.

24 Stock and Watson (2002b) address this issue by regressing the predictor series back onto the factors and grouping predictors into highly correlated groups. Variables in each group are then interpreted as representing a specific economic force.
to achieve significant out-of-sample predictability. Can imposing an economic theory help? To answer this, we consider a dynamic version of the quantity theory of money, building upon Fama (1981, 1982), that links next future inflation $\pi_{t+1}$ to the current growth in real output $(f_{qt})$ and money supply $(m_{t})$.

$$\pi_{t+1} = a_{1}f_{qt} + a_{2}m_{t} + \text{error}_{t+1}. $$

Under this model, forecasts of future inflation may be obtained from observed output growth($q_{t}$) and observed money growth ($m_{t}$). But these quantities may themselves be subject to measurement noise $q_{t} = b_{0} + b_{1}f_{qt} + \omega_{qt}, \quad m_{t} = c_{0} + c_{1}m_{t} + \omega_{mt}.$

How do avoid an errors-in-variables problem? The error-ridden observables may be used as 3PRF theory-proxies for extracting inflation-relevant information from the cross section of macroeconomic predictors $x$ at time $t$ under the assumption that

$$x_{t} = \Phi(f_{qt}, m_{t})' + \omega_{t}. $$

Table 6 reports the results of using theoretically-motivated variables output growth and money growth to directly forecast GDP inflation. Such direct forecasts obtain a 1.3% out-of-sample $R^{2}$, just a bit less than the 2.1% obtained by PCR or 2.8% obtained by FA in Table 4. But when we instead use output growth and money growth as theory-proxies to extract predictive factors from the cross section of macroeconomic predictors, we obtain superior out-of-sample performance with an $R^{2}$ of 7.6%. This improvement is significant according to the Diebold–Mariano–West statistic at the 10% level.

We are essentially using the cross section of predictors to “clean” the noise in the theory-proxies. The resulting forecasts are simple to explain to a policy maker – we forecast tomorrow’s inflation using today’s output and money growth – because basic macroeconomic theory says that those variables determine inflation. We have only used the three-pass regression filter to better triangulate the latent output and money growth factors driving the predictable part of future inflation.

### 6. Conclusion

This paper has introduced a new econometric technique called the three-pass regression filter which is effective for forecasting in a many-predictor environment. The key feature of the 3PRF is its ability to selectively identify the subset of factors that is useful for forecasting a given target variable while discarding factors that are irrelevant for the target but that may be pervasive among predictors. We prove that 3PRF forecasts converge in probability to the infeasible best forecast as $N$ and $T$ simultaneously become large. We also derive the limiting distributions of forecasts and estimated predictive coefficients. We compare our method to principal components regressions following Stock and Watson (2002a) as well as newer forecasting techniques found in Bai and Ng (2008), De Mol et al. (2008) and Doz et al. (2012).

The 3PRF demonstrates strong forecasting performance, and is often superior to alternatives, across a variety of simulation specifications and in empirical applications using macroeconomic and financial data.
afactorstructure. Theorem 7 is a new but basic induction result, and Theorem 8 is new.

Finally, Appendix A.7 is provided to establish the necessity of the relevant proxy assumption, particularly that we need as many relevant proxies as there are relevant factors. We show that it is not generally possible to achieve consistent forecasts using a single 3PRF predictive index when there are multiple relevant factors. In fact, this only obtains in a knife-edge case wherein the relevant factors’ time series variances and relevant loadings’ cross-sectional variances are all equal. Absent this condition, consistency requires as many relevant proxies as there are relevant factors.

A.2. Assumptions

We restate the assumptions here so that the online appendix is self-contained and can be read without referring to assumptions in the main text.

**Assumption 1 (Factor Structure).** The data are generated by the following:

\[ x_t = \phi_0 + \Phi F_t + \epsilon_t \quad y_{t+1} = \beta_0 + \beta F_t + \eta_{t+1} \quad \xi_t = \lambda_0 + \Lambda F_t + \omega_t \]

\[ X = \iota \phi_0 + F \Phi + \epsilon \quad Y = \iota \beta_0 + F \beta + \eta \quad Z = \iota \lambda_0 + FA + \omega \]
Assumption 3

Assumption 2

\[ K = 306 \]

Factors, Loadings and Residuals

Assumption 2 (Factors, Loadings and Residuals). Let \( M < \infty \). For any \( i, s, t \)

1. \( \mathbb{E}[F_i] = \text{M, } T^{-1} \sum_{i=1}^{T} F_i \xrightarrow{T \to \infty} \mu \) and \( T^{-1/2} F_i \xrightarrow{T \to \infty} \Delta_F \)
2. \( \mathbb{E}[\phi_i^4] = M, N^{-1} \sum_{i=1}^{N} \phi_i \xrightarrow{P \to \infty} \phi, N^{-1/2} J_N \Phi \xrightarrow{P \to \infty} \mathcal{P} \) and \( N^{-1} \Phi_j \rightarrow \mathbb{P}_1 \)
3. \( \mathbb{E}(\varepsilon_{it}) = 0, \mathbb{E}[\varepsilon_{it}]^4 \leq M \)
4. \( \mathbb{E}(\omega_{it}) = 0, \mathbb{E}[\omega_{it}]^4 \leq M, T^{-1/2} \sum_{i=1}^{T} \omega_i = O_p(1) \) and \( T^{-1} \Omega_{ij} \xrightarrow{P \to \infty} \Delta_{ij} \)
5. \( \mathbb{E}(\eta_{i,t+1}) = \mathbb{E}(\eta_{i,t+1}|y_{t-1}, F_{t-1}, \ldots) = 0, \mathbb{E}(\eta_{i,t+1}^4) \leq M, \)
   and \( \eta_{i,t+1} \) is independent of \( \phi_i(m) \) and \( \varepsilon_{it} \).

Assumption 3 (Dependence). Let \( \gamma(x) \) denote the mth element of \( x \). For \( \gamma < \infty \) and any \( i, j, t, s, m, n_2 \)

1. \( \mathbb{E}(\varepsilon_{it} \varepsilon_{jt}) = \sigma_{ij,t}, |\sigma_{ij,t}| \leq \hat{\sigma}_j \) and \( |\sigma_{ij,t}| \leq \tau_{it} \), and
   (a) \( N^{-1} \sum_{i=1}^{N} \hat{\sigma}_j \leq M \) (b) \( (N^{-1} \sum_{i=1}^{N} |\sigma_{ij,t}|) \leq M \)
   (b) \( T^{-1} \sum_{t=1}^{T} \tau_{it} \leq M \) (d) \( N^{-1/2} \sum_{i=1}^{N} |\sigma_{ij,t}| \leq M \)
2. \( \mathbb{E}[N^{-1/2} T^{-1/2} \sum_{t=1}^{T} \sum_{i=1}^{N} [\varepsilon_{it} \varepsilon_{jt} - \mathbb{E}(\varepsilon_{it} \varepsilon_{jt})]^2] \leq M \)
3. \( \mathbb{E}[T^{-1/2} \sum_{t=1}^{T} F_t(m_1 \alpha_t(m_2)]^2 \leq M \)
4. \( \mathbb{E}[T^{-1/2} \sum_{t=1}^{T} \alpha_t(m_1) \varepsilon_{it}]^2 \leq M. \)

\( N_{100}, T = 100, K = 0 \)

\( N_{500}, T = 500, K = 1 \)

Fig. A.3. Simulated distribution, \( \hat{\beta} \).

Assumption 4 (Central Limit Theorems). For any \( i, t \)

1. \( N^{-1/2} \sum_{i=1}^{N} \phi_i \varepsilon_{it} \xrightarrow{d} \mathcal{N}(0, \mathbf{\Gamma}_{\phi}) \), where \( \mathbf{\Gamma}_{\phi} = \text{plim}_{N \to \infty} N^{-1} \sum_{i=1}^{N} \mathbb{E}[\phi_i \varepsilon_{it} \varepsilon_{it}] \)
2. \( T^{-1/2} \sum_{t=1}^{T} \eta_{i,t+1} \xrightarrow{d} \mathcal{N}(0, \mathbf{\Gamma}_{\eta}) \), where \( \mathbf{\Gamma}_{\eta} = \text{plim}_{T \to \infty} T^{-1} \sum_{t=1}^{T} \mathbb{E}[\eta_{i,t+1}^2] \)
3. \( T^{-1/2} \sum_{t=1}^{T} \eta_{i,t+1} \xrightarrow{d} \mathcal{N}(0, \mathbf{\Gamma}_{\eta}) \), where \( \mathbf{\Gamma}_{\eta} = \text{plim}_{T \to \infty} T^{-1} \sum_{t=1}^{T} \mathbb{E}[\eta_{i,t+1}^2] > 0. \)

Assumption 5 (Normalization). \( \mathcal{P} = I, \mathcal{P}_1 = 0 \) and \( \Delta_F \) is diagonal, positive definite, and each diagonal element is unique.

Assumption 6 (Relevant Proxies). \( \mathbf{\Lambda} = [\mathbf{A}_{\phi}, \mathbf{0}] \) and \( \mathbf{A}_{\phi} \) is nonsingular.

A.3. Auxiliary lemmas

The following lemma collects basic results for various sums of products of the random variables appearing in our factor system. It repeatedly uses Cauchy–Schwarz and follows arguments appearing in Bai and Ng (2002), Stock and Watson (2002a) and Bai (2003).

Lemma 1. Let Assumptions 1–4 hold. Then for all \( s, t, i, m, m_1, m_2 \)

1. \( \mathbb{E}[(NT)^{-1/2} \sum_{i=1}^{N} F_t(m) (\varepsilon_{it} \varepsilon_{it})^2] \leq M \)
2. \( \mathbb{E}[(NT)^{-1/2} \sum_{i=1}^{N} \alpha_t(m) (\varepsilon_{it} \varepsilon_{it})^2] \leq M \)
3. \( N^{-1/2} T^{-1/2} \sum_{i=1}^{N} \alpha_t = O_p(1) \)
4. \( T^{-1/2} \sum_{i=1}^{N} \eta_{i,t+1} = O_p(1) \)
5. \( T^{-1/2} \sum_{i=1}^{N} \varepsilon_{it} = O_p(1) \)
6. \( N^{-1/2} T^{-1/2} \sum_{i=1}^{N} \eta_{i,t+1} = O_p(1) \)
7. \( N^{-1/2} T^{-1/2} \sum_{i=1}^{N} \phi_i(m_1) \varepsilon_{it} F_t(m_2) = O_p(1) \)
8. \( N^{-1/2} T^{-1/2} \sum_{i=1}^{N} \phi_i(m_1) \varepsilon_{it} \alpha_t(m_2) = O_p(1) \)

\( N_{500}, T = 500, K = 1 \)
The stochastic order is understood to hold as $N, T \to \infty$ and $\delta_{NT} \equiv \min(\sqrt{N}, \sqrt{T})$.

**Proof.** Item 1: Note that

$$
\mathbb{E} \left[ \left( NT \right)^{-1/2} \sum_{i,t} F_t(m) \left[ \epsilon_{it} - \sigma_{it} \right] \right]^2 
\leq \sum_{i,t} \mathbb{E} \left[ F_t(m) \left( \epsilon_{it} - \sigma_{it} \right) \right]^2 
\leq \max_{i,t} \mathbb{E} \left[ F_t(m) \mathbb{E} \left[ \left( NT \right)^{-1/2} \sum_{i,t} \left( \epsilon_{it} - \sigma_{it} \right) \right]^2 \right].
$$

By Assumptions 2.1 and 3.2. The same argument applies to Item 2 using Assumptions 2.4, 3.1.

Item 3: The first part follows from

$$
\mathbb{E} \left[ N^{-1/2} T^{-1/2} \sum_{i,t} \epsilon_{it} \right]^2 = N^{-1/2} T^{-1/2} \sum_{i,t} \sigma_{it}^2 \leq N^{-1/2} T^{-1/2} \sum_{i,t,j} \sigma_{it,j}^2 \leq M \quad \text{by Assumption 3.1.}
$$

The second and third parts of Item 3 follow similar rationale.

Item 4: follows from

$$
\mathbb{E} \left[ T^{-1/2} \sum_{i,t} \eta_{it+1} \right] = T^{-1/2} \sum_{t} \mathbb{E} [\eta_{it+1}] = O_p(1) \quad \text{by Assumption 2.5.}
$$

Item 5: Note that

$$
\mathbb{E} \left[ T^{-1/2} \sum_{i,t} \epsilon_{it} \eta_{it+1} \right] = T^{-1/2} \sum_{i,t} \sigma_{it} \mathbb{E} \left[ \eta_{it+1} \right] \leq T^{-1} \sum_{t} \mathbb{E} [\eta_{it+1}] \sigma^\prime \leq O_p(1) \quad \text{by Assumptions 2.5 and 3.1.}
$$

Item 6: Note that

$$
\mathbb{E} \left[ N^{-1/2} T^{-1/2} \sum_{i,t} \epsilon_{it} \eta_{it+1} \right] = N^{-1/2} T^{-1/2} \sum_{i,t} \sigma_{it} \mathbb{E} \left[ \eta_{it+1} \right] \leq T^{-1} \sum_{t} \mathbb{E} [\eta_{it+1}^2] N^{-1/2} \sum_{t} \sigma_{it}^2 \leq O_p(1) \quad \text{by Assumptions 2.5 and 3.1.}
$$

Item 7 is bounded by

$$
\mathbb{E} \left[ N^{-1/2} T^{-1/2} \sum_{i,t} \phi_i(m_1) \phi_i(m_2) \epsilon_{it} \right] = O_p(1) \quad \text{by Assumptions 2.2 and 4.3.}
$$

Item 8 follows the same rationale as Assumptions 2.2 and 3.4.

Item 9: Note that

$$
\mathbb{E} \left[ N^{-1/2} T^{-1/2} \sum_{i,t} \phi_i(m) \phi_i(m) \epsilon_{it} \right] = N^{-1} T^{-1} \sum_{i,t,j} \mathbb{E} \left[ \phi_i(m) \phi_j(m) \epsilon_{it} \epsilon_{jt} \right] \quad \text{since } \mathbb{E} [\eta_{it+1}] = 0 \text{ for } t \neq s,
$$

which is in turn equal to

$$
T^{-1} \sum_{t} \mathbb{E} [\eta_{it+1}] \mathbb{E} \left[ \left( N^{-1/2} \sum_{i,t} \phi_i(m) \epsilon_{it} \right)^2 \right].
$$

By Assumption 2.5. That this expression is $O_p(1)$ follows from Assumptions 2.5 and 4.1.

Item 10: $N^{-1/2} T^{-1/2} \sum_{i,t} \epsilon_{it} \sigma_{it} \leq T^{-1/2} \sum_{i,t} \sigma_{it}^2 \leq O_p(1)$ by Assumptions 3.2 and 3.1.

Item 11: By Item 10 and Assumption 2.5,

$$
N^{-1/2} T^{-1/2} \sum_{i,t} \epsilon_{it} \sigma_{it} \leq \left( T^{-1/2} \sum_{i,t} \sigma_{it}^2 \right)^{1/2} \leq O_p(1).
$$

Item 12: First, we have

$$
N^{-1/2} T^{-1/2} \sum_{i,t} F_t(m) \epsilon_{it} \sigma_{it} \leq N^{-1/2} \left( N^{-1/2} T^{-1/2} \sum_{i,t} F_t(m) \epsilon_{it} \right) \leq O_p(1).
$$

By Lemma 1 the first term is $O_p(N^{-1/2})$. Because $\mathbb{E} \left[ N^{-1} \sum_{i,t} F_t(m) \sigma_{it} \right] \leq N^{-1} \max \mathbb{E} [F_t(m)] \sum_{i,t} \sigma_{it}$, $O_p(1)$ by Assumption 3.1, the second term is $O_p(T^{-1/2})$. The same argument applies to Item 13 using Item 2.

Item 14: By Assumption 4.3 and Item 5, 

$$
N^{-1/2} T^{-1/2} \sum_{i,t} F_t(m) \epsilon_{it} \eta_{it+1} \leq (N^{-1} \sum \left( N^{-1/2} T^{-1/2} \sum F_t(m) \epsilon_{it+1} \right)^2)^{1/2}
$$

The same argument applies to Item 15 using Assumption 3.4 and Item 5.

The following result builds on the previous lemma. It identifies finite-dimensional matrices that appear in the expression for the 3PRF, and then looks to find the stochastic order of any generic element of the matrix.

**Lemma 2.** Let Assumptions 1–4 hold. Then

1. $T^{-1/2} F_j \omega = O_p(1)$
2. $T^{-1/2} F_j T_j = O_p(1)$
3. $T^{-1/2} \epsilon_j \eta_j = O_p(1)$
4. $N^{-1/2} \epsilon_j T_j \Phi = O_p(1)$
5. $N^{-1/2} \Phi_j \epsilon_j \epsilon_j \Phi = O_p(\delta_{NT})$
6. $N^{-1/2} \Phi_j \epsilon_j \eta_j = O_p(1)$
7. $N^{-1/2} \Phi_j \epsilon_j \eta_j = O_p(1)$
8. $N^{-1/2} \eta_j \epsilon_j \Phi_j \epsilon_j = O_p(\delta_{NT})$
9. $N^{-1/2} \eta_j \epsilon_j \Phi_j \epsilon_j = O_p(\delta_{NT})$
10. $N^{-1/2} \eta_j \epsilon_j \Phi_j \epsilon_j = O_p(\delta_{NT})$

The stochastic order is understood to hold as $N, T \to \infty$, stochastic orders of matrices are understood to apply to each entry, and $\delta_{NT} \equiv \min(\sqrt{N}, \sqrt{T})$.

**Proof.** Item 1: $T^{-1/2} F_j \omega = T^{-1/2} \sum F_t \omega_t - (T^{-1} \sum_f F_t) (T^{-1} \sum_i \omega_t) = O_p(1)$ by Assumptions 2.1, 2.4 and 3.3.

Item 2: $T^{-1/2} F_j T_j = T^{-1/2} \sum F_t \eta_t - (T^{-1} \sum_f F_t) (T^{-1} \sum_i \eta_t) = O_p(1)$ by Lemma 1.4 and Assumptions 2.1 and 4.2.

Item 3: Follows directly from Lemma 1.5, 1.6 and Assumption 2.3.

Item 4 has mth element $N^{-1/2} \sum_i \epsilon_i \phi_i(m) - (N^{-1/2} \sum_i \phi_i(m)) (N^{-1} \sum_i \phi_i(m)) = O_p(1)$ by Assumption 2.2, 2.3, 4.1 and Lemma 1.3.

Item 5 is a $K \times K$ matrix with generic $(m_1, m_2)$ element:

$$
N^{-1/2} \sum_{i,j,t} \phi_i(m_1) F_t(m_2) \epsilon_{it} - N^{-2} T^{-2} \sum_{i,j,t} \phi_i(m_1) F_t(m_2) \epsilon_{jt} = 5.1 - 5.11 - 5.11 + 5.14.
$$

5.1 = $O_p(T^{-1/2})$ by Lemma 1.7.

5.11 = $O_p(T^{-1/2})$ since $N^{-1} \sum_i \phi_i(m_1) = O_p(1)$ by Assumption 2.2 and $N^{-1} \sum_i \epsilon_i (T^{-1/2} \sum F_t(m_2) \epsilon_{jt}) = O_p(1)$ by Assumption 4.3.

5.11 = $O_p(N^{-1/2})$ since $N^{-1} \sum_i \phi_i(m_1) = O_p(1)$ by Assumption 2.1 and $T^{-1} \sum F_t(m_2) \epsilon_{jt} = O_p(1)$ by Assumption 4.1. For the following items in this lemma’s proof we use the argument here and in Item 5II without further elaboration except to change the referenced assumption or lemma items.

5.14 = $O_p(T^{-1/2} N^{-1/2})$ by Assumptions 2.1, 2.2 and Lemma 1.3.

Summing these terms, Item 5 is $O_p(\delta_{NT}^{-1})$

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27 The web appendix rearranges this and following items to clearly show the terms.
Item 6 is a $K \times L$ matrix with generic $(m_1, m_2)$ element
\[
N^{-1/2} \sum_{i,t} \phi_i(m_1) \alpha(i) (m_2) e_{it} = N^{-1/2} \sum_{i,t} \phi_i(m_1) \alpha(i) (m_2) e_{it}
\]
\[= N^{-1/2} \sum_{i,t} \phi_i(m_1) \alpha(i) (m_2) e_{it} + N^{-2/3} \sum_{i,t} \phi_i(m_1) \alpha(i) (m_2) e_{it} = 6.I - 6.II - 6.III + 6.IV.
\]
6.I = $O_p(1)$ by Lemma 1.8.
6.II = $O_p(1)$ by Assumption 2.2, 3.4.
6.III = $O_p(N^{-1/2})$ by Assumption 2.4, 4.1.
6.IV = $O_p, (T^{-1/2}N^{-1/2})$ by Assumption 2.2, 2.4 and Lemma 1.3.
Summing these terms, Item 6 is $O_p(1)$.

Item 7 has generic mth element
\[
N^{-1/2} \sum_{i,t} \phi_i(m_1) \alpha(i) (m_2) e_{it} = N^{-1/2} \sum_{i,t} \phi_i(m_1) \alpha(i) (m_2) e_{it} + N^{-3/2} \sum_{i,t} \phi_i(m_1) \alpha(i) (m_2) e_{it} = 7.I - 7.II - 7.III + 7.IV.
\]
7.I = $O_p(1)$ by Lemma 1.9.
7.II = $O_p(1)$ by Assumption 4.1 and Lemma 1.4.
7.III = $O_p(1)$ by Assumption 2.2 and Lemma 1.6.
7.IV = $O_p(T^{-1/2})$ by Assumption 2.2 and Lemmas 1.3 and 1.4.
Summing these terms, Item 7 is $O_p(1)$.

Item 8 is $K \times L$ matrix with generic $(m_1, m_2)$ element
\[
N^{-1/2} \sum_{i,t} \phi_i(m_1) \alpha(i) (m_2) e_{it} = N^{-1/2} \sum_{i,t} \phi_i(m_1) \alpha(i) (m_2) e_{it} + N^{-3/2} \sum_{i,t} \phi_i(m_1) \alpha(i) (m_2) e_{it} = 8.I - \cdots - 8.VIII.
\]
8.I = $T^{-1/2}N^{-1/2} \sum_{i,t} \phi_i(m_1) \alpha(i) (m_2) e_{it} = T^{-1/2}N^{-1/2} \sum_{i,t} \phi_i(m_1) \alpha(i) (m_2) e_{it} = O_p(T^{-1/2})$ by Assumption 4.3.
8.II = $O_p(\delta_{\eta}^{-1})$ by Assumption 2.1 and Lemma 1.12. Item 8.III is identical.
8.IV = $O_p(\delta_{\eta}^{-1})$ by Assumption 2.1 and Lemma 1.10.
8.V = $O_p(T^{-1/2})$ by Assumption 4.3.
8.VI = $O_p(N^{-3/2}T^{-1/2})$ by Assumptions 2.1, 4.3 and Lemma 1.3.
Item 8.VII is identical.
8.VIII = $O_p(N^{-1/2}T^{-1/2})$ by Assumption 2.1 and Lemma 1.3.
Summing these terms, we have Item 8 is $O_p(\delta_{\eta}^{-1})$.

Items 9 and 10 follow the same argument as Item 8 but replace where appropriate $w_i(m)$ for $F_i(m)$, Lemma 1.13 for 1.12 and Assumption 3.4 for 4.3. Substituting this way implies Items 9, 10 are no larger than $O_p(\delta_{\eta}^{-1})$.

Item 11 has generic mth element given by
\[
N^{-1/2} \sum_{i,t} \phi_i(m_1) \alpha(i) (m_2) e_{it} = N^{-1/2} \sum_{i,t} \phi_i(m_1) \alpha(i) (m_2) e_{it} + N^{-3/2} \sum_{i,t} \phi_i(m_1) \alpha(i) (m_2) e_{it} = 11.I - 11.II - 11.III + 11.IV.
\]
11.I = $O_p(\delta_{\eta}^{-1})$ by Lemma 1.12.
11.II = $O_p(N^{-1/2})$ by Assumption 4.3 and Lemma 1.3.
11.III = $O_p(\delta_{\eta}^{-1})$ by Assumption 2.1 and Lemma 1.10.
11.IV = $O_p(N^{-1/2})$ by Assumption 2.1 and Lemma 1.3.
Summing these terms, we have Item 11 is $O_p(\delta_{\eta}^{-1})$.

Item 12 follows nearly the same argument as Item 11, but replaces $w_i(m)$ for $F_i(m)$ and Assumptions 3.4 for 4.3. Substituting this way implies that Item 12 is $O_p(\delta_{\eta}^{-1})$.

Item 13 has mth element
\[
N^{-1/2} \sum_{i,t} \phi_i(m_1) \alpha(i) (m_2) e_{it} = N^{-1/2} \sum_{i,t} \phi_i(m_1) \alpha(i) (m_2) e_{it} + N^{-3/2} \sum_{i,t} \phi_i(m_1) \alpha(i) (m_2) e_{it} = 13.I - \cdots - 13.VIII.
\]
13.I = $O_p(T^{-1/2})$ by Lemma 1.14.
13.II = $O_p(T^{-1/2}N^{-1/2})$ by Lemmas 1.12 and 1.4.
13.III = $O_p(\delta_{\eta}^{-1})$ by Assumption 2.1 and Lemma 1.11.
13.IV = $O_p(T^{-1/2}N^{-1/2})$ by Assumption 2.1 and Lemmas 1.3 and 1.4.
13.V = $O_p(N^{-3/2}T^{-1/2})$ by Assumption 4.3 and Lemma 1.6.
13.VI = $O_p(N^{-1/2}T^{-1})$ by Assumption 4.3 and Lemmas 1.3 and 1.4.
13.VII = $O_p(N^{-1/2}T^{-1})$ by Assumption 2.1 and Lemmas 1.3 and 1.6.
13.VIII = $O_p(N^{-1/2}T^{-1})$ by Assumption 2.1 and Lemmas 1.3 and 1.4.

Summing these terms, Item 13 is $O_p(\delta_{\eta}^{-1})$.

Item 14 follows the same argument as Item 13 replacing Lemma 1.15 for 1.14, Lemma 1.13 for 1.12 and Assumption 3.4 for 4.3. Substituting this way implies that Item 14 is $O_p(\delta_{\eta}^{-1})$.

A.4. Probability limits and forecast consistency

This lemma finds the probability limit for our factor estimator $\hat{F}$. It expands out this expression to find terms involving $X, Z, y$ that can then be expressed using Assumption 1 as matrices appearing in Lemma 2.

**Lemma 3.** Let Assumptions 1–4 hold. Then the probability limits of $\hat{F}$ and $\hat{F}_i$ are
\[
\hat{F} \xrightarrow{p}{T \to \infty} \left(\Lambda \Delta \Lambda + \Delta_{\omega} \right)^{-1} \Lambda \Delta \Phi
\]
and

\[ \hat{F}_t \xrightarrow{p} \left( \Lambda \Delta_F \Lambda' + \Delta_\omega \right) \left( \Lambda \Delta_F \mathcal{P} \Delta_F \Lambda' \right)^{-1} \left( \Lambda \Delta_F P_1 + \Lambda \Delta_F \mathcal{P} F_t \right). \]

**Proof.** From Eq. (2), for any \( t \) the second stage 3PRF regression coefficient is

\[ \hat{F}_t = T^{-1} Z' F_t Z \left( N^{-1} T^{-2} Z' X_j X' j' Z \right)^{-1} N^{-1} T^{-1} Z' X_j X' j' \]

\[ = \hat{F}_t \hat{F}_t \xrightarrow{p} \hat{F}_t \]

We handle each of these three terms individually.

\[ \hat{F}_A = T^{-1} Z' F_t Z \]

\[ = \Lambda \left( T^{-1} F' F_t F \right) \Lambda' + \Lambda \left( T^{-1} F' F_t \omega \right) \]

\[ + \left( T^{-1} \omega' F_t \right) \Lambda' \]

\[ \xrightarrow{p} \Lambda \Delta_F \Lambda' + \Delta_\omega. \]

\[ \hat{F}_B = N^{-1} T^{-2} Z' j' X_j X' j' Z \]

\[ = \Lambda \left( T^{-1} F' j' F \right) \left( N^{-1} \Phi j' j' \Phi \right) \left( T^{-1} F' F_t F \right) \Lambda' \]

\[ + \Lambda \left( T^{-1} F' j' F \right) \left( N^{-1} \Phi j' j' \Phi \right) \left( T^{-1} F' F_t \omega \right) \]

\[ + \left( T^{-1} \omega' F_t \right) \left( N^{-1} \Phi j' j' \Phi \right) \Lambda' \]

\[ \xrightarrow{p} \Lambda \Delta_F \mathcal{P} \Delta_F \Lambda' \]

\[ \hat{F}_{C,t} = N^{-1} T^{-1} Z' j_t X_j \]  \quad (*A.1*)

\[ = \Lambda \left( T^{-1} F' j_t F \right) \left( N^{-1} \Phi j_t j_t \Phi \right) \Lambda' \]

\[ + \Lambda \left( T^{-1} F' j_t F \right) \left( N^{-1} \Phi j_t j_t \Phi \right) \Lambda' \]

\[ + \left( T^{-1} \omega' j_t \right) \left( N^{-1} \Phi j_t j_t \Phi \right) \Lambda' \]

\[ \xrightarrow{p} \Lambda \Delta_F P_1 + \Lambda \Delta_F \mathcal{P} F_t. \]

Each convergence result follows from Lemma 2 and Assumptions 1–4. The final result is obtained via the continuous mapping theorem. The result for \( \Phi \) proceeds similarly, using the result above for \( \hat{F}_t \) and the fact that \( \text{plim}_{N,T \to \infty} T^{-1} Z' F_t X = \Delta \Lambda \Phi \) using Lemma 2. \( \square \)

This lemma finds the probability limit for our factor estimator \( \hat{\beta} \). It expands out this expression to find terms involving \( X, Z, \), \( y \), that can then be expressed using Assumption 1 as matrices appearing in Lemma 2.

**Lemma 4.** Let Assumptions 1–4 hold. Then the probability limit of estimated third stage predictive coefficients \( \hat{\beta} \) is

\[ \hat{\beta} = \xrightarrow{p} \left( \Lambda \Delta_F \Lambda' + \Delta_\omega \right)^{-1} \Lambda \Delta_F \mathcal{P} \Delta_F \Lambda' \]

\[ \times \left( \Lambda \Delta_F \mathcal{P} \Delta_F \Lambda' \right)^{-1} \Lambda \Delta_F \mathcal{P} \Delta_F \beta. \]  \quad (A.2)

**Proof.** From Eq. (3), the third stage 3PRF regression coefficient is

\[ \hat{\beta} = \left( T^{-1} Z' F_t Z \right)^{-1} N^{-1} T^{-2} Z' j_t X_j X' j' Z \]

\[ \times \left( N^{-1} T^{-2} Z' j_t X_j X' j' Z \right)^{-1} N^{-1} T^{-2} Z' j_t X_j X' j' y \]

\[ = \hat{\beta}_1 \hat{\beta}_2 \hat{\beta}_3. \]

We handle each of these three terms individually. Note that \( \hat{\beta}_1 = \hat{F}_A \) and \( \hat{\beta}_2 = \hat{F}_B \) and these probability limits are established in Lemma 3. The expressions for \( \hat{\beta}_1 \) and \( \hat{\beta}_2 \) are more tedious and require an additional lemma (that holds given Assumptions 1–4) which we place in the web appendix. Therefore we have that

\[ \hat{\beta}_1 = N^{-2} T^{-2} Z' j_t X_j X' j' y \]

\[ \xrightarrow{p} \Lambda \Delta_F \mathcal{P} \Delta_F \Lambda' \]

and

\[ \hat{\beta}_2 = N^{-1} T^{-2} Z' j_t X_j X' j' y \]

\[ \xrightarrow{p} \Lambda \Delta_F \mathcal{P} \Delta_F \beta. \]

Each convergence result follows from Lemma 2 and Assumptions 1–4. The final result is obtained via the continuous mapping theorem. \( \square \)

This lemma finds the probability limit for our factor estimator \( \hat{y} \), but is immediate from the two preceding proofs.

**Lemma 5.** Let Assumptions 1–3 hold. Then the three-pass regression filter forecast satisfies

\[ \hat{y}_{t+1} \xrightarrow{p} \beta_0 + \mu' \beta + \left( F_t - \mu \right) \mathcal{P} \Delta \Lambda' \]

\[ \times \left[ \Lambda \Delta_F \mathcal{P} \Delta_F \Lambda' \right]^{-1} \Lambda \Delta_F \mathcal{P} \Delta_F \beta. \]  \quad (A.3)

**Proof.** Immediate from Eq. (1) and Lemmas 3 and 4. \( \square \)

This theorem uses the probability limit just found for \( \hat{y} \) and adds the assumption that the proxies are relevant. This allows certain off-diagonal matrices to go to zero, ensuring consistency.

**Theorem 1.** Let Assumptions 1–6 hold. The three-pass regression filter forecast is consistent for the infeasible best forecast, \( \hat{y}_{t+1} \xrightarrow{p} \beta_0 + F_t \beta \).

**Proof.** Given Assumptions 1–3, Lemma 5 holds and we can therefore manipulate (A.3). Partition \( \mathcal{P} \) and \( \Delta \) as

\[ \mathcal{P} = \begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix}, \quad \Delta_F = \begin{bmatrix} \Delta_F_{11} & \Delta_F_{12} \\ \Delta_F_{21} & \Delta_F_{22} \end{bmatrix} \]

such that the block dimensions of \( \mathcal{P} \) and \( \Delta \) coincide. By Assumption 5, the off-diagonal blocks, \( \mathcal{P}_{12} \) and \( \Delta_{F,12} \), are zero. As a result, the first diagonal block of the term \( \Delta_F \mathcal{P} \Delta_F \mathcal{P} \Delta_F \) in Eq. (A.3) is \( \Delta_F_{11} \mathcal{P}_{11} \Delta_F_{11} \mathcal{P}_{11} \Delta_F_{11} \). By Assumption 6, pre- and post-multiplying by \( \Lambda = \left[ \Lambda_f, 0 \right] \) reduces the term in square brackets to \( \Lambda_f \Delta_F_{11} \mathcal{P}_{11} \Delta_F_{11} \mathcal{P}_{11} \Lambda_f \). Similarly, \( \mathcal{P} \Delta_F \Lambda' = \left[ \Lambda_f \mathcal{P}_{11} \Delta_F_{11}, 0 \right] \) and
\[ \Delta \Delta \rho \Delta F = [A_i \Delta F, \rho \Delta F, 1, 0] \] By Assumption 6, \( A_i \) is invertible and therefore the expression for \( \hat{\beta}_{i+1} \) reduces to \( \hat{\beta}_0 + F_i \beta \).

**Corollary 1.** Let Assumptions 1–5 hold. Additionally, assume that there is only one relevant factor. Then the target-proxy three pass regression filter forecaster is consistent for the infeasible best forecast.

**Proof.** It follows directly from previous result by noting that the loading of \( y \) on \( F \) is \( \beta = (\beta_1, 0') \) with \( \beta_1 \neq 0 \). Therefore the target satisfies the condition of Assumption 6.

We consider a generic element of the projection coefficient \( \alpha \) and obtain its probability limit, which boils down to performing matrix algebra.

**Theorem 2.** Let \( \hat{\alpha}_i \) denote the \( i \)th element of \( \hat{\alpha} \), and let Assumptions 1–6 hold. Then for any \( i \),

\[ N \hat{\alpha}_i \xrightarrow{p} \begin{cases} \phi_i - \hat{\phi} \bigg\rvert \beta \end{cases} \]

**Proof.** Rewrite \( \hat{\alpha}_i = S_i \hat{\alpha} \), where \( S_i \) is the \( (1 \times N) \) selector vector with \( i \)th element equal to one and remaining elements zero. Expanding the expression for \( \hat{\alpha} \) in Eq. (4), the first term in \( S_i \hat{\alpha} \) is the \( (1 \times K) \) matrix \( S_j f_j \Phi \), which has probability limit \( (\Phi - \hat{\Phi}) \) as \( N, T \to \infty \). It then follows directly from previous results that

\[ N \hat{\alpha}_i \xrightarrow{p} \begin{cases} \phi_i - \hat{\phi} \bigg\rvert \beta \end{cases} \]

Under Assumptions 5 and 6, this reduces to \( (\phi_i - \hat{\phi}) \bigg\rvert \beta \).

The following lemma finds the probability limit of the predictors “residuals” that are unexplained by the factor estimator \( \hat{F} \) in the limit. Notice that \( \hat{e} \) is consistent for the true idiosyncratic errors (for which cross-sectional dependence is limited by Assumption 3) and a linear combination of the irrelevant factors \( g \) which can be pervasive across predictors. This fact complicates the construction of a consistent estimator for the asymptotic variance of \( \hat{F} \).

**Lemma 6.** Define \( \hat{e} = X - \hat{\phi} \hat{\Phi} - \hat{F} \hat{\Phi} \), where \( \hat{\phi}_0 = T^{-1} \sum_i x_i - \hat{\Phi} \left( T^{-1} \sum_i F_i \right) \). Under Assumptions 1–6, \( \hat{F} \hat{\Phi} \xrightarrow{p} f \Phi_f \) and

\[ \hat{e} \xrightarrow{p} \begin{cases} e + g \Phi' \end{cases} \]

**Proof.** Let \( S_i \) be a \( K \times K \) selector matrix that has ones in the first \( K_f \) main diagonal positions and zeros elsewhere. The fact that

\[ \hat{F} \hat{\Phi}' \xrightarrow{p} \begin{cases} \Lambda \Delta F P + \Lambda \Delta F P F' \bigg\rvert \Lambda \Delta F P F' \bigg\rvert -1 \Lambda \Delta F \Phi' \end{cases} \]

follows directly from Lemma 3. By Assumptions 5 and 6, this reduces to \( \hat{F}_0 \Phi' = f \Phi_f \), which also implies the stated probability limit of \( \hat{e} \).

The following lemma establishes the asymptotic independence of \( \hat{F} \) and \( \eta_{i+1} \), which is used to find the asymptotic distribution of \( \hat{\alpha} \).

**Lemma 7.** Under Assumptions 1–4, \( \text{plim}_{N,T \to \infty} T^{-1} \sum_t \hat{F}_t \eta_{t+1} = 0 \) for all \( h \).

28 This proof shows that Assumption 5 is stronger than is necessary. All we require is that \( P \) and \( \Delta_F \) are block diagonal.

**Proof.** It suffices to show that \( \text{plim}_{N,T \to \infty} T^{-1} \sum_t \hat{F}_t \eta_{t+1} = 0 \) for all \( h \), and to do so we examine each term in Eq. (A.1). The four terms involving \( \hat{F}_t \) becomes \( o_p(1) \) since each is \( o_p(1) \) by Lemma 2, since they do not possess \( t \) subscripts, and since \( T^{-1} \sum_t \eta_{t+1} = o_p(1) \). By similar rationale, the four terms that are post-multiplied by \( F_t \) are \( o_p(1) \) since \( T^{-1} \sum_t F_t \eta_{t+1} = o_p(1) \) by Assumption 4.3. Two of the remaining terms depend on the expression \( T^{-1} \sum_t (N^{-1} \Phi_j \epsilon_t) \eta_{t+1} \), which is \( o_p(1) \) because

\[ \left| T^{-1} N^{-1} \sum_t \phi_i \epsilon_t \eta_{t+1} \right| \leq N^{-1/2} \left\{ T^{-1} \sum_t \left( N^{-1/2} \sum_i \phi_i \epsilon_t \right)^2 \right\}^{1/2} \] \[ \times \left( T^{-1} \sum_t \eta_{t+1}^2 \right)^{1/2} = o_p(1). \]

The last two remaining terms depend on \( T^{-1} \sum_t (N^{-1} \Phi_j \epsilon_t) \eta_{t+1} \), which is \( o_p(1) \) following the same argument used to prove Lemma 2.14.

**A.5. Asymptotic distributions**

**Lemma 8.** Under Assumptions 1–4, as \( N, T \to \infty \) we have

\[ N^{-1} T^{-3/2} Z_j X_j X_j' \to \mathcal{N} \left( 0, A \Delta F \mathcal{P} F' \Delta F A' \right). \]

**Proof.**

\[ N^{-1} T^{-2} Z_j X_j X_j' \to N^{-1} T^{-2} \Delta F \Phi_f \Phi_f' \Phi_f' \Phi_f' \to \mathcal{N} \left( 0, A \Delta F \mathcal{P} F' \Delta F A' \right). \]

As \( N, T \to \infty \), the first term is dominant and the stated asymptotic distribution obtains by Lemma 2 and Assumption 4.2.

**Theorem 3.** Under Assumptions 1–6, as \( N, T \to \infty \) we have

\[ \sqrt{T N} \left( \hat{\alpha}_i - \alpha_i \right) \xrightarrow{d} \mathcal{N} \left( 0, 1 \right) \]

where \( \alpha_i \) is the \( i \)th diagonal element of \( \Sigma_{\alpha} \left( \hat{\alpha} \right) \), \( \Sigma_{\alpha} \left( \hat{\alpha} \right) = \Omega_{\alpha} \left( \sum_i \hat{\eta}_{i+1}^2 (X_i - \hat{\Xi}) (X_i - \hat{\Xi})' \Omega_{\alpha} \right. \), \( \hat{\eta}_{i+1} \) is the estimated 3PRF forecast error and

\[ \Omega_{\alpha} = J_f \left( \frac{1}{T} S_{XZ} \right) \left( \frac{1}{T^2 N^2} W_{XZ} S_{XX} W_{XZ} \right)^{-1} \left( \frac{1}{T N} W_{XZ} \right). \]

**Proof.** Given the definition of \( \hat{\alpha}_i \), note that

\[ N \hat{\alpha}_i - N \hat{\alpha}_i \xrightarrow{d} \mathcal{N} \left( 0, 1 \right) \]

The asymptotic distribution and consistent variance estimator follow directly from Lemma 8 and previously derived limits, Assumptions 5 and 6, and noting that \( \hat{\eta}_{i+1} = \eta_{i+1} + o_p(1) \) by Theorem 1.
Theorem 4. Under Assumptions 1–6, as $N, T \to \infty$ we have
$$\sqrt{T} (\hat{y}_{t+1} - y_{t+1}) \xrightarrow{d} \mathcal{N}(0, \Sigma_{\beta})$$
where $y_{t+1} = \bar{y} + X_t \hat{c}_{\beta}$ and $Q_\beta^2$ is the $t$th diagonal element of $\frac{1}{T} X_t \text{Var}(\hat{a}) X_t'$. 

Proof. The result follows directly from Theorems 2 and 3. Note that the theorem may be restated replacing $\bar{y}_{t+1}$ with $E_\epsilon y_{t+1}$ since the argument leading up to Theorem 1 implies that $\sqrt{T} \bar{y}_{t+1} \xrightarrow{p} E_\epsilon y_{t+1}$. By Slutsky’s theorem convergence in distribution follows, yielding the theorem statement in the paper’s text. □

Theorem 5. Under Assumptions 1–6, as $N, T \to \infty$ we have
$$\sqrt{T} \left( \hat{\beta} - G_\beta \beta \right) \xrightarrow{d} \mathcal{N}(0, \Sigma_\beta)$$
where $\Sigma_\beta = \Sigma^{-1}_\beta T_{\beta} \Sigma^{-1}_\epsilon$ and $\Sigma_\epsilon = \Lambda \Delta_F \Lambda' + \Delta_\omega$. Furthermore,
$$\hat{\text{Var}}(\hat{\beta}) = (T^{-1} \hat{J}_t \hat{F})^{-1} T^{-1} \sum \hat{\eta}^2_{t+1} (\hat{F} - \mu) (\hat{F} - \mu)' (T^{-1} \hat{J}_t \hat{F})^{-1}$$
is a consistent estimator of $\Sigma_\beta$. 

Proof. Define $G_\beta = \beta^{-1} \hat{\beta} \hat{\beta}^{-1} \left( N^{-1} T^{-2} Z_t' X_t X_t' F_t \right)$. The asymptotic distribution follows directly from Lemma 8 noting that
$$\hat{\beta} - G_\beta \beta = \beta^{-1} \beta \hat{\beta}^{-1} \left( N^{-1} T^{-2} Z_t' X_t X_t' F_t \right).$$
The asymptotic covariance matrix (before employing Assumptions 5 and 6) is $\Sigma_\beta = \Psi_\beta T_{\beta} \Psi_\beta'$, where $\Psi_\beta = \Sigma^{-1} \Lambda \Delta_F \Lambda' + \Delta_\omega$. This expression follows from Lemma 8 and the probability limits derived in the proof of Lemma 4, Assumptions 5 and 6 together with the argument in the proof of Theorem 1 reduces $\Sigma_\beta$ to the stated form.

To show consistency of $\hat{\text{Var}}(\hat{\beta})$, note that $\sqrt{T} \left( \hat{\beta} - G_\beta \beta \right) = (T^{-1} \hat{J}_t \hat{F})^{-1} T^{-1} \hat{J}_t \eta - \hat{F} \eta \hat{F}'$, which implies that the asymptotic variance of $\hat{\beta}$ is equal to the probability limit of
$$(T^{-1} \hat{J}_t \hat{F})^{-1} \left( T^{-1} \hat{J}_t \eta \right) \left( T^{-1} \hat{J}_t \hat{F} \right)^{-1}.$$

(A.4)

Assumption 2.5 and Lemma 7 imply that $\lim_{T \to \infty} T^{-1} \hat{J}_t \eta \hat{J}_t' \hat{F} = \lim_{n \to \infty} n^{-1} \sum_{n=1} F_n - \mu) (\hat{F} - \mu)'$. By Theorem 1, $\hat{\eta}_{t+1} = \hat{\eta}_{t+1} + o_p(1)$, which implies that $\hat{\text{Var}}(\hat{\beta})$ and (A.4) share the same probability limit, therefore $\hat{\text{Var}}(\hat{\beta})$ is a consistent estimator of $\Sigma_\beta$. □

Lemma 9. Under Assumptions 1–4, as $N, T \to \infty$ we have

(i) if $\sqrt{T}/T \to 0$, then for every $t$
$$N^{-1/2} T^{-1} Z_t' X_t \epsilon_t \xrightarrow{d} \mathcal{N} \left( 0, \Lambda \Delta_F' \Gamma_{\phi_t} \Delta_F \Lambda' \right)$$

(ii) if $\lim \inf \sqrt{T}/T \geq t \geq 0$, then
$$N^{-1} Z_t' X_t \epsilon_t = O_p(1).$$

Proof. From Lemma 2 we have
$$N^{-1} T^{-1} Z_t' X_t \epsilon_t = \hat{F}_{t,t} - N^{-1} T^{-1} Z_t' X_t \epsilon_t$$
$$= \Lambda (T^{-1} F_t' F_t (N^{-1} \Phi \epsilon_t \epsilon_t') + \Lambda (N^{-1} T^{-1} F_t' \epsilon_t \epsilon_t') + (T^{-1} \omega_t' F_t (N^{-1} \Phi \epsilon_t \epsilon_t') + (N^{-1}) \omega_t' F_t (N^{-1} \Phi \epsilon_t \epsilon_t') + (T^{-1} \omega_t' F_t (N^{-1} \Phi \epsilon_t \epsilon_t') + O_p((N^{-1} T^{-1} - 1/2) + O_p((N^{-1} T^{-1} - 1/2) + O_p((N^{-1} T^{-1} - 1/2).$$

When $\sqrt{N}/T \to 0$, the first term determines the limiting distribution, in which case result (i) obtains by Assumption 4.1.

When $\lim \inf \sqrt{T}/T \geq t \geq 0$, we have $T (N^{-1} T^{-1} Z_t' X_t \epsilon_t) = O_p(1)$ since $\lim \inf T/\sqrt{T} \leq 1/t < \infty$. □

Define
$$H_0 = \hat{F}_{t,t}^{-1} N^{-1} T^{-1} Z_t' X_t \epsilon_t \Phi_0$$
and
$$H = \hat{F}_{t,t}^{-1} N^{-1} T^{-1} Z_t' X_t \Phi.$$ 

Theorem 6. Under Assumptions 1–6, as $N, T \to \infty$ we have for every $t$

(i) if $\sqrt{T}/T \to 0$, then
$$\sqrt{T} \left( \hat{F}_t - (H_0 + HF) \right) \xrightarrow{d} \mathcal{N} \left( 0, \Sigma_t \right)$$

(ii) if $\lim \inf \sqrt{T}/T \geq t \geq 0$, then
$$T \left( \hat{F}_t - (H_0 + HF) \right) = O_p(1)$$

where $\Sigma_t = \left( \Lambda \Delta_\Phi \Lambda' + \Delta_\omega \right) \left( \Lambda \Delta_\Phi \Lambda' \right)^{-1} \Lambda \Delta_\Phi \Gamma_{\theta_t} \Delta_\Phi \Lambda' \left( \Lambda \Delta_\Phi \Lambda' \right)^{-1} \Lambda \Delta_\Phi \Lambda'. □$

A.6. Automatic proxy selection

Theorem 7. Let Assumptions 1–5 hold with the exception of Assumptions 2.4, 3.3, 3.4. Then the L-auto-proxy three pass regression filter of $y$ automatically satisfies Assumptions 2.4, 3.3, 3.4, 6 when $L = K_t$. As a result, the L-auto-proxy is consistent and asymptotically normal according to Theorems 1 and 4.

Proof. We begin by showing that Assumption 6 is generally satisfied. If $K_t = 1$, Assumption 6 is satisfied by using $y$ as the proxy (see Corollary 1). For $K_t > 1$, we proceed by induction to show that the automatic proxy selection algorithm constructs a set of proxies that satisfies Assumption 6. In particular, we wish to show that the automatically-selected proxies have a loading matrix on relevant factors (A) that is full rank, and that their loadings on irrelevant factors are zero. We use superscript $(k)$ to denote the use of $k$ automatic proxies.

Denote the 1-automatic-proxy 3PREF forecast by $\tilde{y}^{(1)}$. We have from Eq. (1) that
$$r^{(1)} = y - \tilde{y}^{(1)} = \eta + F \beta - \hat{F}^{(1)} \hat{\beta}$$
$$= F (\beta - \Phi (k \beta)) + \eta + \epsilon (\beta - \eta),$$
where $\Omega^{(1)} = \hat{J}_t X_t' Z (Z_t' X_t, \hat{X}_t, \hat{X}_t)' (Z_t' X_t, \hat{X}_t, \hat{X}_t)' \Lambda_{\hat{J}} X_t X_t' F_t$. For $r^{(1)}$, $\Omega^{(1)}$ is constructed based on $Z = y$. Recalling that $\beta =$
(\beta_r, 0')', it follows that \( y \) has zero covariance with irrelevant factors, so \( g(\theta) \) also has zero covariance with irrelevant factors and therefore \( r^{(1)} \) has population loadings of zero on irrelevant factors. To see this, note that irrelevant factors can be represented as \( \beta F(0, I)' \), where the zero matrix is \( K \times K \) and the identity matrix is dimension \( K \). This, together with Assumption 2.5, 4.3, implies that the cross product matrix \( 0, I)' \) is zero in expectation.

The induction step proceeds as follows. By hypothesis, suppose we have \( k < K \) automatically-selected proxies with factor loadings \( \{ \alpha_{r,k} \} \), where \( \alpha_{r,k} = k \times K \) and full row rank. The residual from the \( k \)-automatic-proxy 3PRF forecast is \( \mathbf{r} = y - \hat{y}^{(k)} \), which has zero population covariance with irrelevant factors by the same argument given in the \( k = 1 \) case. It is left to show that the \( r^{(k)} \) taking on relevant factors is linearly independent of the rows of \( \alpha_{r,k} \).

To this end, note that these relevant-factor loadings take the form \( \beta_r - \Phi_i \Omega_i \beta_i \), where \( f = \mathcal{F} S_{k} \), and \( S_{k} = [I, 0]' \) is the matrix that selects the first \( K \) columns of the matrix \( \Omega_i \) that multiplies (the form of this loading matrix follows again from \( \beta = [\beta_r, 0'] \)). Also note that as part of the induction hypothesis, \( \Omega_i \) is constructed based on \( Z = (r^{(1)}, \ldots, r^{(k-1)} \).

Next, project \( r^{(k)} \)'s relevant factors onto the column space of \( \alpha_{r,k} \). The residual's loading vector is linearly independent of \( \alpha_{r,k} \) if the difference between it and its projection on \( \alpha_{r,k} \) is non-zero. Calculating this difference, we find \( (I - \alpha_{r,k} \alpha_{r,k}^{-1}) \alpha_{r,k} = [I - \Phi_i \Omega_i \beta_i] \beta_i = [I - \Phi_i \Omega_i \beta_i] \beta_i \).

Because \( (I - \Phi_i \Omega_i \beta_i) \neq 0 \) with probability one, this difference is zero only when \( \alpha_{r,k} \alpha_{r,k}^{-1} \) is non-zero. But the induction hypothesis ensures that this is not the case since \( k > K \). Therefore the difference between the \( r^{(k)} \)'s loading vector and its projection onto the column space of \( \alpha_{r,k} \) is nonzero, thus its loading vector is linearly independent of the rows of \( \alpha_{r,k} \). Therefore we have constructed proxies that satisfy Assumption 6.

We next show that the \( L \)-automatic-proxy 3PRF satisfies Assumptions 2.4, 3.3 and 3.4 when the remaining parts of Assumptions 1–6 hold. Each automatic proxy is a forecast error, so \( z_t = y_{t+1} - \hat{y}_{t+1} \), where the forecast \( \hat{y}_{t+1} \) is a linear combination of predictors. By similar limiting arguments as those leading up to Theorem 1, this linear combination can be generically expressed as \( N^{-1/2} a' x_i \), where \( a = \mathcal{O}(1) \). We can rewrite an automatic proxy \( z_t \) (suppressing constants) as \( z_t = b_f \xi + \alpha _w \eta \). For \( \alpha _w \eta \) to be independent and can be handled separately. By Assumption 2.5, 4.2, the \( \eta \) component directly satisfies Assumptions 2.4, 3.3, 3.4.

Assumption 2.4 also requires \( \mathbb{E} [\sum \alpha _w \eta] = 0 \), and \( \mathbb{E} \left[ \frac{1}{N} \sum \alpha _w \eta \right] = 0 \), which are satisfied by Assumptions 2.3 and 3.2. Assumption 3.3 requires \( \mathbb{E} \left[ \frac{1}{\sqrt{N}} \sum \alpha _w \eta \right] = 0 \), which is satisfied by Assumption 3.2.

Together these results imply that the \( L \)-automatic-proxy satisfies the conditions of Theorems 1 and 4 when \( L = K_f \).

The following proposition simply shows that the 3PRF is the constrained least squares estimator of a projection of \( y \) onto \( X \). The body of the text interprets this constraint as the assumption that the relevant factor space is spanned by one's choice of proxy variables.

**Theorem 8.** The three-pass regression filter's implied \( N \)-dimensional predictive coefficient, \( \hat{\alpha} \), is the solution to

\[
\arg \min \| y - \alpha X \| \\
\text{subject to } (I - W_{X} S_{W_{X}})^{-1} W_{X} \alpha = 0.
\]

**Proof.** By the Frisch–Vaugh–Lovell Theorem, the value of \( \alpha \) that solves this problem is the same as the value that solves the least squares problem for \( \| \mathbf{y} - \mathbf{I} \mathbf{X} \alpha \| \). From Amemiya (1985, Section 1.4), the estimate of \( \alpha \) that minimizes the sum of squared residuals \( \mathbf{y} - \mathbf{I} \mathbf{X} \alpha \) subject to the constraint \( Q ' \alpha = c \) is found by

\[
\mathbf{R} (\mathbf{R} S_{X} \mathbf{R})^{-1} \mathbf{R} X' s_{W} + [I - R (\mathbf{R} S_{X} \mathbf{R})^{-1} \mathbf{R} S_{X}] Q (Q' Q)^{-1} c
\]

for \( R \) such that \( R Q = 0 \) and \( Q' R \) is nonsingular. In our case, \( c = 0 \) and \( Q = I - W_{X} (s_{W} W_{X})^{-1} W_{X} \), hence we can let \( R = W_{X} \) and the result follows.

**A.7. Relevant proxies and relevant factors**

This section explores whether, given our normalization assumptions, it is possible in general to reformulate the multi-factor system as a one-factor system, and achieve consistent forecasts with the 3PRF using a single automatically selected proxy (that is, the target-proxy 3PRF). The answer is that this is not generally possible. We demonstrate this both algebraically and in simulations. The summary of this section is:

I. There is a knife-edge case (which is ruled out by Assumption 5) in which the target-proxy 3PRF is always consistent regardless of \( K_f \).

II. In the more general case (consistent with Assumption 5) the target-proxy 3PRF is inconsistent for \( K_f > 1 \) but the \( K_f \) automatic-proxy 3PRF is consistent.

To demonstrate points 1 and 2, we begin from our normalization assumptions and show that three necessary conditions for consistency must hold for any rotation of the factor model. Second, we show that in the knife-edge case the target-proxy 3PRF is consistent (routted out in our main development by assumption) but that the general case consistency continues to require as many proxies as there are relevant factors. This remains true when the multi-factor model is reformulated in terms of a single factor. Third, we provide simulation evidence that supports these conclusions.

Heuristically speaking, the main intuition of this section is the following: The 3PRF's consistency requires that the first-pass and second-pass regressions be consistent, which in turn requires that they have no omitted variable bias. For the first pass regression this is satisfied by the assumption that the factors are orthogonal. For the second pass regression, since it is on the loadings this is satisfied only once all the relevant factors have been spanned since we only require that relevant factors' loadings and irrelevant factors' loadings are orthogonal (a simple normalization assumption) and that not each relevant factor's loading is orthogonal to every other (an assumption that is stronger than mere normalization).

**A.7.1. Our original representation**

Our analysis centers on the probability limit given in Lemma 5. For simplicity, we assume in this appendix that \( y, x, f \) and \( \phi \) are mean zero, \( K_f = \mathcal{O}(f) > 1 \), suppress time subscripts, and assume

\[
\mathbb{E} (f' f) = \Delta_f = \begin{bmatrix} \Delta_f & \Delta_f \\ \Delta_f & \Delta_f \end{bmatrix}, \quad \mathbb{E} (f' \phi) = 0, \quad \mathbb{E} (\phi' \phi) = 0.
\]

The points we make in this simpler case transfer directly to the model described in the main text. The probability limit of \( \hat{y} \) may therefore be rewritten as

\[
\hat{y} \overset{p}{\underset{T \rightarrow \infty}{\rightarrow}} \mathcal{F} \mathcal{R} \Delta_f \mathcal{A'} \left[ \mathcal{A} \mathcal{R} \mathcal{F} \mathcal{F} \mathcal{R} \mathcal{F} \mathcal{A'} \right]^{-1} \mathcal{A} \mathcal{R} \mathcal{F} \mathcal{R} \mathcal{F} \mathcal{A'} \mathbf{B}.
\]

By inspection, consistency requires three conditions to ensure that the coefficient vector post-multiplying \( \mathcal{F} \) in (A.6) reduces to \( (\beta_r', 0)' \). These conditions are:

1. The factors \( f \) and \( \phi \) are independent.
2. The factors \( f \) and \( \phi \) are orthogonal.
3. The factors \( f \) and \( \phi \) are unrelated to the error process.
1. $A = [A_f \ 0]$ (Relevant proxies)
2. $\Delta_{ig} = 0$ (Relevant factors orthogonal to irrelevant factors)
3. $\beta_{fg} = 0$ (Relevant factors loadings orthogonal to irrelevant factors loadings).

To see that these are necessary, first note that condition 1 implies that $P' \Delta_f A'$ reduces to

$$
\begin{bmatrix}
\beta_{f1} & \beta_{f2} & \ldots & \beta_{fn}
\end{bmatrix}
\begin{bmatrix}
A_{f1} & A_{f2} & \ldots & A_{fn}
\end{bmatrix}
\begin{bmatrix}
\beta_{g1} & \beta_{g2} & \ldots & \beta_{gm}
\end{bmatrix}
\begin{bmatrix}
\beta_{g1} & \beta_{g2} & \ldots & \beta_{gm}
\end{bmatrix}
\begin{bmatrix}
\Delta_{f1} & \Delta_{f2} & \ldots & \Delta_{fn}
\end{bmatrix}
\begin{bmatrix}
\Delta_{g1} & \Delta_{g2} & \ldots & \Delta_{gm}
\end{bmatrix}
$$

(A.7)

Since the same matrix $[(A f A')^{-1} A f A']$ post-multiplies both of these rows, we can here determine the necessity of conditions 2 and 3. The bottom row of (A.7) must be 0 for the irrelevant factors to drop out. Conditions 2 and 3 achieve this while avoiding degeneracy of the underlying factors and factor loadings.

Given necessary conditions 1–3, we have that $\tilde{y}$ is reduced to

$$f' \beta \tilde{y} = f' \beta_f \tilde{y}$$

Consistency requires that (A.8) reduces to $f' \beta_f$. We are now in a position to identify the knife-edge and general cases. The knife-edge case occurs when $\beta_f \tilde{y} = \sigma f$ and $\beta_f = \beta_f$, for positive scalar $\sigma$. In this case (A.8) becomes

$$\sigma \beta_f \left[ \sigma^2 \beta_f' \beta_f \right]^{-1} \sigma \beta_f' \beta_f = \beta_f$$

The target-proxy 3PRF is consistent even though there are $K_f > 1$ relevant factors in the original system.

In the general case, we only assume $P_f \Delta_f$, $A_f$ are invertible (so that $P_f \Delta_f$ need not be an equivariance matrix). In this case (A.8) reduces to $f' \beta_f$. The key condition here is the invertibility of these matrices, which requires using $K_f > 1$ relevant proxies (obtainable by the auto-proxy algorithm). This is the paper’s main result.

Recalling the discussion in Stock and Watson (2002a) and Section 2.2, it is quite natural that the final condition required for consistency involves both the factor (time-series) variances and the (cross-sectional) variances of the factor loadings: This is the nature of identification in factor models. The general point is that requirements for identification and consistent estimation of factor models requires assumptions regarding both factors and loadings. By convention we assume that factors are orthogonal to one another. The loadings can then be rotated in relation to the factor space we have assumed, but not all rotations are observationally-equivalent once we have pinned down the factor space.

### A.7.2. A One-factor representation of the multi-factor system

Let us rewrite the factor system by condensing multiple relevant factors into a single relevant factor:

$$h = \beta_f' f.$$ 

In addition, we can rotate the original factors so that the first factor $h$ is orthogonal to all others. Let this rotation be achieved by some matrix $M$ such that

$$m = M f.$$ 

The new formulation therefore satisfies

$$y = h + \eta,$$

$$x = \Psi_h m + \Psi_m m + \Psi_g g + \epsilon,$$

and

$$\Lambda = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$ 

Now $h$ is the single relevant factor while $(m' \ g)'$ are the irrelevant factors. We have represented the system such that first two necessary conditions for consistency are satisfied. We now show that the third necessary condition will not be satisfied in general.

Let us write the loadings in this rotated system $(\Psi_h, \Psi_m, \Psi_g)$ in terms of the loadings in the original system $(\Phi_f, \Phi_g)$. Because $\mathbb{E}(h m')$, $\mathbb{E}(hg)$, $\mathbb{E}(mg')$ are all zero, we recover

$$\mathbb{E}(x - \Psi_h h, h) = 0 \Rightarrow \Psi_h = \frac{1}{\beta_f' \Delta_f \beta_f} \Phi_f \Delta_f \beta_f.$$

$$\mathbb{E}(x - \Psi_m m, m) = 0 \Rightarrow \Psi_m = \Phi_f \Delta_f (M' \Delta_f M)^{-1}$$

$$\mathbb{E}(x - \Psi_g g, g') = 0 \Rightarrow \Psi_g = \Phi_g.$$ 

The covariance matrix of loadings is therefore

$$NN' \sum_{i=1}^{N} \begin{bmatrix} \psi_{h,i} \\ \psi_{m,i} \\ \psi_{g,i} \end{bmatrix}$$

and the third necessary condition is determined by whether or not the matrix

$$NN' \sum_{i=1}^{N} \begin{bmatrix} \psi_{h,i} \\ \psi_{m,i} \\ \psi_{g,i} \end{bmatrix} = 0$$

equals zero in the limit. The second element $\psi_{h,i}$ is in light of (A.9). However, in the general case, $\beta_f' \Delta_f \beta_f M = 0$ even though (A.9) holds and the third necessary condition cannot generally be satisfied in this rewritten system.

### A.7.3. Simulation study

We now run a Monte Carlo to demonstrate that, when there are multiple relevant factors, a target-proxy achieves the infeasible best only when the knife-edge case holds. Our simulation design uses the following:

$$y = f + \eta,$$

$$x = \begin{bmatrix} f \\ g \end{bmatrix} \Phi' + \epsilon$$

where $\epsilon$ is $K_f \times 1$ ones vector, $g(T \times K_g)$, $\Phi(N \times K_f + K_g)$, $\eta(T \times 1)$, and $\epsilon(T \times N)$ i.i.d standard normal, and $f(T \times K_f)$ is i.i.d normal with standard deviation $\sigma_f$. The infeasible best forecast for this system is $f_1$. We use six factors, three relevant and three irrelevant ($K_f = K_g = 3$) and consider different values for $N$, $T$, and $\sigma_f$. We consider $N = T = 200$ and $N = T = 2000$. We use an identity covariance matrix for factor loadings ($P = I$) and consider two values for $\sigma_f$: a knife-edge (equivariant) case $[1 \ 1 \ 1]$ and a more general (non-equivariant) case $[0.5 \ 1 \ 2]$. Table A.1 lends simulation support to our algebraic proof. We focus on in-sample results since out-of-sample results are qualitatively similar.

In the knife-edge case the target-proxy 3PRF appears consistent. For $N = T = 2000$ the correlation between the 3PRF forecast and the infeasible best forecast is 0.993, and their relative $R^2$ is 0.9901.
For $N = T = 200$ these numbers are lower, but that is attributable to the smaller sample.

In the general case the target-proxy 3PRF appears inconsistent. The relative $R^2$ is 0.8425 for $N = T = 200$ and 0.8586 for $N = T = 2000$; the correlation is 0.9169 for $N = T = 200$ and 0.9241 for $N = T = 2000$. This agreement across the two sample sizes is strongly suggestive that the inconsistency is not a small sample issue, but rather holds in large $N$, for which 2000 is a good approximation. Furthermore, the relative $R^2$ increases notably as we move to 2 auto-proxies: 0.9736 for $N = T = 200$ and 0.9762 for $N = T = 2000$. Once we have 3 auto-proxies (as our theorem states) the simulation evidence suggests that the 3PRF is consistent. The relative $R^2$ is 0.9938 for $N = T = 200$ and 0.9983 for $N = T = 2000$.

### A.8. Accuracy of asymptotic theory in finite samples

Our first experiment evaluates the accuracy of finite sample approximations based on the asymptotic distributions we have derived. We examine the distributions of predictive coefficient estimates as well as the forecasts themselves. For each Monte Carlo draw, we first compute the estimates $\hat{y}$, $\hat{\alpha}$ and $\hat{\beta}$. Then we standardize each estimate in accordance with Theorems 3–5 by subtracting off the mean adjustment term and dividing by the respective asymptotic standard error estimate. According to the theory presented in Section 2, these standardized estimates should follow a standard normal distribution for large $N$ and $T$.

For each estimator (corresponding to Figs. A.1–A.3) we plot the distribution of standardized estimates across simulations (solid line) versus the standard normal pdf (dashed line). The four panels of each figure correspond to $N = 100$, $T = 100$ and $N = 500$, $T = 500$ in the cases that (i) there is a single relevant factor and (ii) there is one relevant and one irrelevant factor. Factors, factor loadings and shocks are drawn from a standard normal distribution. The predictive loading on the relevant factor is set to one (that is, the infeasible best $R^2$ is set equal to 50%). We simulate 5000 samples for each set of parameter values.

These results show that the standard normal distribution successfully describes the finite sample behavior of these estimates, consistent with the results in Section 2. In all cases but one we fail to reject the standard normal null hypothesis for standardized estimates. The exception occurs for $\hat{\beta}$ when $N = 100$ and $T = 100$, which demonstrates a minor small sample bias (Fig. A.3, upper right). This bias vanishes when the sample size increases (Fig. A.3, lower right). The simulated coverage rates of a 0.95 confidence interval for $\hat{y}_{t+1}$ are also well behaved. For $N = 100$ and $T = 100$ the simulated coverage is 0.95 when there is no irrelevant factor and 0.94 when an irrelevant factor exists. For $N = 500$ and $T = 500$ the simulated coverage is 0.947 and 0.949 and, respectively. Altogether, simulations provide evidence that the 3PRF accurately estimates the infeasible best forecasts and predictive coefficients, and that its theoretical asymptotic distributions accurately approximate the finite sample distributions for 3PRF estimates.

### A.9. Information criterion Monte Carlo

In Section 4.2 we discuss an information criterion for selecting the number of predictive indices when using the auto-proxy 3PRF. Our degrees of freedom calculation uses the “Trace of the Krylov Representation” method of Kramer and Sugiyama (2011). In particular, when using $m$ 3PRF automatic proxies, the degrees of freedom are given by

$$
\text{DoF}(m) = 1 + \sum_{j=1}^{m} c_j \text{trace}(K^j) - \sum_{l=1}^{m} \text{trace}(K^l t_l) + (y - \hat{y}_m) \sum_{j=1}^{m} K^j v_j + m
$$

where $K = XX'$, $c_j$ are elements of the vector $c = B^{-1} Ty$, $B$ is a Krylov basis decomposition, $T$ is the matrix of PLS factor estimate vectors $t_j$, and $v_j$ are columns of the matrix $T(B^{-1})'$. The BIC is then calculated as $\sum_{l=1}^{m} \sum_{j=1}^{m} (y_l - \hat{y}_{m,l})^2 / T + \log(T) \sigma^2 \text{DoF}(m) / T$ where $\sigma^2 = \sum_{l=1}^{m} (y_l - \hat{y}_{m,l})^2 / (T - \text{DoF}(m))$. We refer readers to Kramer and Sugiyama (2011) for additional details.

Table A.2 studies the accuracy of the information criterion in the simulation specifications of Table 3, and compares how 3PRF1...

<table>
<thead>
<tr>
<th># auto proxies:</th>
<th>In-Sample</th>
<th>Out-of-Sample</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 2 3</td>
<td>1 2 3</td>
</tr>
<tr>
<td>$N = T = 200$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_f = [1 1 1]$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho(\hat{y}, f_1)$</td>
<td>0.9607</td>
<td>0.9316</td>
</tr>
<tr>
<td>$\rho(\hat{y}, f_2)$</td>
<td>0.9678</td>
<td>0.9649</td>
</tr>
<tr>
<td>$\rho(\hat{y}, f_3)$</td>
<td>0.8425</td>
<td>0.8820</td>
</tr>
<tr>
<td>$N = T = 2000$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_f = [1 1 1]$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho(\hat{y}, f_1)$</td>
<td>0.9901</td>
<td>0.9850</td>
</tr>
<tr>
<td>$\rho(\hat{y}, f_2)$</td>
<td>0.9930</td>
<td>0.9929</td>
</tr>
<tr>
<td>$\rho(\hat{y}, f_3)$</td>
<td>0.8786</td>
<td>0.8877</td>
</tr>
</tbody>
</table>

Notes: $\rho(\hat{y}, f_1)$ denotes the average ratio of 3PRF $R^2$ to the infeasible best $R^2$. $\rho(\hat{y}, f_1)$ gives the average time series correlation between the 3PRF forecast and the infeasible best forecast.

To see the table, please refer to the document.
The table reports performance in terms of out-of-sample forecast percentage $R^2$ for the 3PRF based on the actual number of relevant factors (column 3PRFIC) or the number of factors selected by an information criterion (column 3PRFIC). For the IC version, we report the average number of factors chosen by the criterion (column #IC). The data generating processes are described in Section 4 and Table 3.

The table also includes some pathological cases in which 3PRFIC outperforms 3PRF. This occurs when both the irrelevant factors and the idiosyncrasies are strongly serially correlated, but the relevant factors are quickly mean reverting. In this case, the first 3PRF factor is corrupted by persistent irrelevant information, and additional 3PRF factors allow the procedure to pick up residual relevant information missed by the first factor.

### A.10. Partial least squares

Like the three-pass regression filter and principal components, partial least squares (PLS) constructs forecasting indices as linear combinations of the underlying predictors. These predictive indices are referred to as “directions” in the language of PLS. The PLS forecast based on the first $K$ PLS directions, $\hat{y}_{kt}^{(K)}$, is constructed according to the following algorithm (as stated in Hastie et al. (2009)):

1. Standardize each $x_i$ to have mean zero and variance one by setting $\hat{x}_i = \frac{x_i - \bar{x}_i}{\hat{\sigma}(\bar{x}_i)}$, $i = 1, \ldots, N$.

2. Set $\hat{y}_{it} = \bar{y}_t$ and $\hat{x}_{i}^{(0)} = \hat{x}_i$, $i = 1, \ldots, N$.

3. For $k = 1, 2, \ldots, K$
   (a) $u_{i}^{(k)} = \sum_{i=1}^{N} \phi_{ki} \hat{x}_{i}^{(k-1)}$, where $\phi_{ki} = \text{Cov}(\hat{x}_{i}^{(k-1)}, y)$.
   (b) $\hat{\beta}_k = \frac{\text{Cov}(u_{i}^{(k)}, y)}{\text{Var}(u_{i})}$.
   (c) $\hat{y}_{it}^{(k)} = \hat{y}_{it}^{(k-1)} + \hat{\beta}_k u_{i}^{(k)}$.
   (d) Orthogonalize each $\hat{x}_{i}^{(k-1)}$ with respect to $u_{i}$: $\hat{x}_{i}^{(k)} = \hat{x}_{i}^{(k-1)} - \left( \frac{\text{Cov}(u_{i}, \hat{x}_{i}^{(k-1)})}{\text{Var}(u_{i})} \right) u_{i}$.
   $i = 1, 2, \ldots, N$.

### A.11. Empirical procedures

The recursive out-of-sample forecasting procedure for macroeconomic data following Bai and Ng (2008) and Stock and Watson (2012) is as follows. Before forecasting each target, we first transform the data by partialing the target and predictors with respect to a constant and four lags of the target. To construct a time $t + 1$ out-of-sample forecast, consider the data known at time $t$: $y_{i} = \{y_t, x_{t} z_{t-1}, x_{t-1} z_{t-2}, \ldots\}$. Calculate either the 3PRF’s three passes or PCR’s eigenvalue decomposition on $y_t$. For the target-proxy 3PRF: the first pass regressions are of $x_{t} z_{t-1}$ on $y_t$ and a constant for $t = 1, 2, \ldots, T$, separately run for each $i = 1, 2, \ldots, N$, yielding $\hat{\phi}_i$; the second pass regression is of $x_{t-1}$ on $\hat{\phi}_i$ and a constant for $i = 1, 2, \ldots, N$, separately run for each $T = 1, 2, \ldots, T$, yielding $\hat{\beta}_i$; the third pass regression is of $y_t$ on $\hat{\beta}_i$ and a constant for $t = 1, 2, \ldots, T$, yielding $\hat{\delta}_0, \hat{\beta}$. Then the out-of-sample forecast is constructed as $\hat{\delta}_0 + \hat{\beta}$. 

### Table A.2

<table>
<thead>
<tr>
<th>$\rho_1$</th>
<th>$\rho_2$</th>
<th>$\alpha$</th>
<th>$\delta$</th>
<th>$N = T = 100$</th>
<th>$N = T = 200$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>3PRF1</td>
<td>3PRFIC</td>
</tr>
<tr>
<td>0.3</td>
<td>0.9</td>
<td>0</td>
<td>0</td>
<td>26.3</td>
<td>18.1</td>
</tr>
<tr>
<td>0.3</td>
<td>0.9</td>
<td>0.3</td>
<td>0.9</td>
<td>25.2</td>
<td>19.1</td>
</tr>
<tr>
<td>0.3</td>
<td>0.9</td>
<td>0.9</td>
<td>0</td>
<td>18.0</td>
<td>12.0</td>
</tr>
<tr>
<td>0.9</td>
<td>0.3</td>
<td>0</td>
<td>0</td>
<td>30.8</td>
<td>20.9</td>
</tr>
<tr>
<td>0.9</td>
<td>0.3</td>
<td>0.3</td>
<td>0.9</td>
<td>31.7</td>
<td>21.6</td>
</tr>
<tr>
<td>0.9</td>
<td>0.3</td>
<td>0.9</td>
<td>0</td>
<td>30.7</td>
<td>21.9</td>
</tr>
<tr>
<td>0.9</td>
<td>0.3</td>
<td>0.9</td>
<td>0</td>
<td>27.8</td>
<td>17.3</td>
</tr>
</tbody>
</table>

| 0.3    | 0.9    | 0       | 0      | 17.7  | 10.9   | 1.8 | 24.4  | 17.6   | 2.1 |
| 0.3    | 0.9    | 0.3     | 0.9    | 9.4   | 6.3    | 1.5 | 25.6  | 18.9   | 2.2 |
| 0.3    | 0.9    | 0.9     | 0      | 27.5  | 19.0   | 1.1 | 37.5  | 29.0   | 1.4 |
| 0.9    | 0.3    | 0.3     | 0.9    | 23.7  | 14.3   | 1.5 | 35.6  | 27.1   | 1.2 |
| 0.9    | 0.3    | 0.9     | 0      | 27.8  | 23.5   | 1.5 | 33.0  | 26.9   | 1.2 |
| 0.9    | 0.3    | 0.9     | 0      | 26.7  | 21.0   | 1.5 | 31.3  | 27.8   | 1.3 |

| 0.3    | 0.9    | 0       | 0      | 8.4   | −16.5  | 1.2 | 24.6  | 19.4   | 1.0 |
| 0.3    | 0.9    | 0.3     | 0.9    | 7.7   | −11.9  | 1.3 | 23.6  | 18.9   | 1.0 |
| 0.3    | 0.9    | 0.9     | 0      | 3.3   | −10.9  | 1.9 | 19.7  | 17.1   | 1.0 |
| 0.3    | 0.9    | 0.9     | 0      | −0.9  | 2.1    | 3.1 | 10.7  | 20.9   | 2.8 |
| 0.3    | 0.9    | 0.9     | 0      | −1.5  | 3.5    | 3.4 | 10.1  | 21.5   | 3.0 |
| 0.9    | 0.3    | 0       | 0      | 17.6  | −5.8   | 1.1 | 29.0  | 20.2   | 1.0 |
| 0.9    | 0.3    | 0.3     | 0.9    | 11.6  | −4.4   | 1.6 | 24.6  | 19.3   | 1.0 |
| 0.9    | 0.3    | 0.9     | 0      | 24.0  | 17.5   | 1.4 | 26.5  | 26.4   | 1.2 |
| 0.9    | 0.3    | 0.9     | 0      | 22.2  | 17.8   | 1.5 | 24.9  | 25.3   | 1.3 |

Notes: The table reports performance in terms of out-of-sample forecast percentage $R^2$ for the 3PRF based on the actual number of relevant factors (column 3PRFIC) or the number of factors selected by an information criterion (column 3PRFIC). For the IC version, we report the average number of factors chosen by the criterion (column #IC). The data generating processes are described in Section 4 and Table 3.

For most parameter configurations the average number of factors chosen ranges from 1.1 to 3.4.

In larger sample (T, N = 200), 3PRF performance is much less affected by relying on the information criterion to select the number of factors. The drop in $R^2$ versus 3PRF tends to be small, and for most parameter configurations the average number of predictors chosen is 1.0.

The table also includes some pathological cases in which 3PRFIC outperforms 3PRF. This occurs when both the irrelevant factors and the idiosyncrasies are strongly serially correlated, but the relevant factors are quickly mean reverting. In this case, the first 3PRF factor is corrupted by persistent irrelevant information, and additional 3PRF factors allow the procedure to pick up residual relevant information missed by the first factor.
For financial data, we do not partial the target or predictors as a first step. But the remaining steps of the recursive out-of-sample forecasting procedure are done, to ensure that a time $t$ forecast (of the time $t + 1$ realization) uses only information (and estimates) available at time $t$.

A.12 Portfolio data construction

We construct portfolio-level log price–dividend ratios from the CRSP monthly stock file using data on prices and returns with and without dividends. Twenty-five portfolios (five-by-five sorts) are formed on the basis of underlying firms’ market equity and book-to-market ratio, mimicking the methodology of Fama and French (1992). Characteristics for year $t$ are constructed as follows. Market equity is price multiplied by common shares outstanding at the end of December. Book-to-market is the ratio of book equity in year $t − 1$ to market equity at the end of year $t$. Book equity is calculated from the Compustat file as book value of stockholders’ equity plus balance sheet deferred taxes and investment tax credit (if available) minus book value of preferred stock (in that order). When Compustat data is unavailable, we use Moody’s book equity data (if available) or par value of preferred stock (in that order). When Compustat preferred stock is defined as either redemption, liquidation (if available) minus book value of preferred stock. Book value of equity plus balancesheet deferred taxes and investment tax credit is calculated from the Compustat file as book value of stockholders’ equity plus market-to-book ratio, mimicking the methodology of Fama and French formed on the basis of underlying firms’ market equity and book-to-market ratio.

We perform sorts on the basis of book-to-market to ensure uniformity in characteristics across portfolios in both dimensions. Stock sorts for characteristic-based portfolio assignments are performed using equally-spaced quantiles as breakpoints to avoid excessively lopsided allocations of firms to portfolios. That is, for a $K$-bin sort, portfolio breakpoints are set equal to the $\{\frac{100}{K}, \frac{200}{K}, \ldots, \frac{(K−1)100}{K}\}$ quantiles of a given characteristic.

References

Bai, J., Ng, S., 2002. Determining the number of factors in approximate factor models. Econometrica 70 (1), 191–221.